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Théophile T. Azomahou^a, Raouf Boucekkine^{b2}, Bity Diene^c

^aUNU-MERIT, Maastricht University, The Netherlands

^bCORE, Department of Economics, UCL, Belgium

^cBETA-Theme, Université Louis Pasteur, Strasbourg 1, France

April 14, 2008

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JEL codes: O41, I20, J10

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² Corresponding author. Department of economics and CORE, Université catholique de Louvain, and department of economics, University of Glasgow. Place Montesquieu, 3, Louvain-la-Neuve (Belgium). Tel. +32 10 47 38 48, Fax +32 10 47 39 45, E-mail: boucekkine@core.ucl.ac.be.

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1 Introduction

Increase in life expectancy is often associated with higher economic growth. A 1998 World Bank study showed that life expectancy displays a strong tendency to improve with per capita income, ranging from as low as 37 years in Sierra Leone to as high as 77 in Costa Rica, more than 12 times richer. Bhargava et al. (2001) used a parametric panel data specification and found that the dynamics of demography indicators such as lagged life expectancy variable is a significant predictor of economic growth. Chakraborty (2004) developed a theoretical model and checked its empirical consistency using a parametric cross-country regression. The author found that life expectancy has a strong and positive effect on capital accumulation.

Yet the insightful work of Kelley and Schmidt (1994, 1995) also clearly highlighted that the relationship between economic growth and longevity is far from linear. In their celebrated 1995 paper, they examined the economic-demographic correlations within parametric panel data framework (89 countries over three decades: 1960-1970, 1970-1980, 1980-1990). As in Brander and Dowrick (1994) and Barlow (1994), the authors also attempted to explicitly incorporate the dynamics of demographic effects by including both contemporaneous and lagged effects of crude birth and death rates. They principally found that demographic processes matter considerably in economic development but in a complex way. Indeed, for the 30-year panel, they observed that population growth has a negative impact on economic growth. Moreover, an increase in the crude birth rate reduces economic growth (eventually through the channel of a negative dependency rate on saving), while a decrease in the crude death rate increases economic growth. For the latter, it seems that in less developed countries, mortality reduction is clustered in the younger and/or working ages. In contrast, in the developed countries, such gains occurred in the retired cohort. Kelley and Schmidt (1995) concluded that population growth is not all good or all bad for economic growth: both elements coexist.

Since the publication of the highly influential paper of Kelley and Schmidt (1995), the relationship between demographic variables and economic development has been the subject of plenty of papers in the economic growth and economic demography literatures. In particular, Boucekkine et al. (2002, 2003, 2004), Boucekkine et al. (2007) have already built and tested some models which effectively deliver the same message: the relationship between economic development and longevity is nonlinear and essentially non-monotonic. All these models are based on a single growth engine, human capital accumulation. The associated mechanisms is the following: (i) a higher life expectancy is likely to lengthen the schooling time, thus inducing a better education and better conditions for economic development; (ii) but at the same time, the fraction of people who did their schooling a long time ago will rise, implying a negative effect on growth, which may be even worse if we account for voluntary retirement. Overall, the effect of increasing longevity on growth is ambiguous, and much less simple than the common view. Another paper taking the human capital accumulation approach is Kalemli-Ozcan et al. (2000), who reached similar conclusions in a much more stylized models.

In this paper, we provide a further and closer empirical and theoretical analysis of the relationship between life expectancy and economic growth relying on an historical panel data with long time series. More precisely, we use an historical panel data for 18 countries spanning

over 1820-2005. We believe that such an historical panel is particularly interesting to capture the relationship between life expectancy and economic development (GDP). In particular, the data include some historical periods with very low values for both life expectancy and economic growth, which may entail a period-specific kind of relationship between the two variables.

In order to have the most flexible and neutral statistical framework, we use a nonparametric approach where no a priori parametric functional form is assumed. Most empirical studies in the literature are generally based on *ad hoc* parametric specifications with little attention paid to model robustness; yet different parametric specifications can lead to significantly different conclusions, and a functional misspecification problem is likely to occur. The main result of our work is to uncover a new kind of nonlinearity in the relationship between life expectancy and economic growth. In particular, while the economic growth rate is found to be increasing in life expectancy, this relationship is strictly convex for low values of life expectancy, and concave for high values of this variable.

Such findings cannot be reproduced within human vintage capital models of the Boucekkine *et al.* type. In such models, the obtained relationship is typically hump-shaped under certain conditions. We therefore propose an alternative theory which captures quite naturally the convex-concave nature of the relationship between longevity and economic growth. To this end, we move to simpler models with physical capital accumulation. We study how life-cycle behaviour combined with a physical capital accumulation engine yielding endogenous growth as in Romer's learning-by-investing (1986) can generate the convex-concave shape. We show that the outcome crucially relies on the demographic structure assumed and on the way age-profiles of certain key economic variables like earnings are modelled. In the perpetual youth model, as depicted in Blanchard (1985), both survival rates and earnings are age-independent. We show that the convex-concave relationship outlined above cannot arise in such a framework. The paper includes two extensions of this basic framework allowing first for age-dependent labor income and then for age-dependent survival probabilities. Age-dependent earnings, in the spirit of Faruquee *et al.* (1997), induce a hump-shaped relationship between life expectancy and economic growth. In contrast, more realistic survival laws *à la* Boucekkine *et al.* (2002) do generate the convex-concave feature identified in the prior empirical investigation. The intuitions behind the results will be clear along the way. In particular, the assumption of age-dependent mortality rates taken in the more realistic modeling is absolutely crucial. Because people with different ages have different lifetimes in such a case, they will have different *effective* planning horizons, and notably different saving decisions. This will be shown to crucially matter in the shape of the relationship between longevity and economic development. Previous contributions merging Blanchard-like structures and physical capital accumulation can be found in Aisa and Pueyo (2004) and Echevarria (2004). None uses the realistic demographic modeling considered in this paper.

The paper is organized as follows. Section 2 presents the empirical framework and results. Section 3 considers the benchmark model merging Blanchard (extended to allow for population growth) and Romer structures. Section 4 introduces the first extension of the benchmark model incorporating age-dependent earnings. Section 5 studies the model with more realistic demographics, that is with age-dependent survival probabilities. Section 6 concludes.

2 An empirical inspection using historical data

The complex dependence of life expectancy on income till a certain threshold where the shape can reverse suggests to model empirically the growth rate of GDP using a flexible nonparametric framework. Furthermore, in our empirical setting, we follow the bulk of the literature but we do not control for all possible determinants for the growth rate of GDP.

Several arguments can be put forward in support of our choice. The first, obvious one, concerns historical data limitations. In this respect, it is important to note that using panel methods that sweeping country effects away allows us to control implicitly for any time invariant determinant. The second obvious and more important point is that, we are not concerned here with obtaining the best predictions for the growth rate of GDP but with the *shape* of the relationship between the latter and life expectancy. In this respect, determinants of the growth rate of GDP which are not correlated with life expectancy become irrelevant. Moreover the impact of determinants which *are* correlated with life expectancy will be captured via life expectancy. Depending on the question asked, this can be seen as a drawback or as an advantage. It is a drawback if we purport to determine the ceteris paribus impact of life expectancy on the growth rate of GDP – but what list of regressors would guarantee this? It is an advantage if we are interested in the global effect of life expectancy, including indirect effects linked with omitted variables.

2.1 The statistical specification

We use a Generalized Additive Model (hereafter GAM) for panel data.¹ Additive models are widely used in theoretical economics and statistics. Deaton and Muellbauer (1980) provides examples in which a separable structure is well designed for analysis and important for interpretability. From statistical viewpoint, the GAM specification has the advantage of avoiding the ‘curse of dimensionality’ which appears in nonparametric regressions when many explanatory variables are accounted for. It also allows to capture non-linearities and heterogeneity in the effect of explanatory variables on the response variable. Moreover, the statistical properties (optimal rate of convergence and asymptotic distribution) of the estimator is well known (see e.g., Stone, 1980). The structure of the model is given by

$$y_{it} = \sum_{j=1}^p f_j(\mathbf{x}_{it}^j) + \mu_i + \varepsilon_{it}, \quad i = 1, \dots, N, \quad t = 1, \dots, T \quad (1)$$

where y_{it} denotes the response variable (here the growth rate of GDP per capita), \mathbf{x}_{it}^j s are j explanatory variables for $j = 1, \dots, p$ (here \mathbf{x} denotes the life expectancy at birth), the f_j are unknown univariate functions to be estimated, μ_i is unobserved individual specific effects. We assume that errors ε_{it} are independent and identically distributed, but no restriction is placed on the temporal variance structure. To account for possible endogeneity, that is, potential feedback from the growth rate of GDP per capita on life expectancy we use the *predeterminedness* assumption:

$$\mathbb{E}(\varepsilon_{it} | \mathbf{x}_{it}, \mathbf{x}_{i,t-1}, \dots, \mathbf{x}_{i1}) = 0,$$

¹See e.g., Hastie and Tibshirani (1990) and Stone (1985) for further details on GAM.

which is fairly much weaker than the *strict exogeneity* assumption

$$\mathbb{E}(\varepsilon_{it}|\mathbf{x}_{i1}, \dots, \mathbf{x}_{iT}) = 0, \quad i = 1, \dots, N, \quad t = 1, \dots, T$$

often made when working with panel data. Now, observe that there are three sources of randomness in the specification (1) which are \mathbf{x}_{it} , μ_i and ε_{it} , and we may want to place less possible restrictions on their joint distribution. Moreover, the possibility to sweep away the unobserved effect μ_i allows us to keep unspecified the joint distribution of \mathbf{x}_{it} and μ_i , and that of μ_i and ε_{it} . Thus, we make no assumption on $\mathbb{E}(\mu_i|\mathbf{x}_{i\ell_1}, \dots, \mathbf{x}_{i\ell_K})$ for any set of dates ℓ_1, \dots, ℓ_K in $\{1, \dots, T\}^2$. The unobserved effect μ_i can be eliminated by differentiating or computing the within transformation. Lagging relation (1) by one period and subtracting yields

$$y_{it} - y_{i,t-1} = \sum_{j=1}^p f_j(\mathbf{x}_{it}^j) - \sum_{j=1}^p f_j(\mathbf{x}_{i,t-1}^j) + \eta_{it}, \quad (2)$$

where $\eta_{it} = \varepsilon_{it} - \varepsilon_{i,t-1}$. A central hypothesis in our framework is the *first difference assumption* (FDA):

$$\mathbb{E}(\eta_{it}|\mathbf{x}_{it}^j, \mathbf{x}_{i,t-1}^j) = 0, \quad i = 1, \dots, N, \quad t = 2, \dots, T$$

which identifies the functions

$$\mathbb{E} \left[y_{it} - y_{i,t-1} | \mathbf{x}_{it}^j, \mathbf{x}_{i,t-1}^j \right] = \sum_{j=1}^p f_j(\mathbf{x}_{it}^j) - \sum_{j=1}^p f_j(\mathbf{x}_{i,t-1}^j), \quad (3)$$

with the norming condition $\mathbb{E}[f_j(\mathbf{x}_{it}^j, \mathbf{x}_{i,t-1}^j)] = 0$, since otherwise there will be free constants in each of the functions. It should be noticed that a special case under which FDA is satisfied is strict exogeneity which drives the within estimator for parametric panel models (see e.g., Wooldridge, 2002). Moreover, in our context, strict exogeneity precludes any feedback from the current value of the growth rate of GDP per capita on future values of life expectancy, which is not a realistic hypothesis. It is also worthwhile noticing that predeterminedness is neither necessary nor sufficient for FDA to hold. It is not sufficient since under predeterminedness alone, we have

$$\mathbb{E}(\eta_{it}|\mathbf{x}_{it}, \mathbf{x}_{i,t-1}, \dots, \mathbf{x}_{i1}) = -\mathbb{E}(\varepsilon_{i,t-1}|\mathbf{x}_{it}, \mathbf{x}_{i,t-1}, \dots, \mathbf{x}_{i1}),$$

which will not be null in general.³ This calls for an extension of predeterminedness yielding (3):

$$\mathbb{E}(\varepsilon_{it}|\mathbf{x}_{i,t+1}, \mathbf{x}_{it}, \mathbf{x}_{i,t-1}, \dots, \mathbf{x}_{i1}) = 0, \quad i = 1, \dots, N, \quad t = 1, \dots, T-1$$

with predeterminedness still holding for $t = T$. In our case, this only precludes feedback from the current value of growth rate of GDP per capita on next year's value of life expectancy, but not on later values. In other words, we allow possible feedback from the current value of growth rate of GDP per capita on values of life expectancy starting from $t+2$, which thus appears as a fairly weak condition compared to the strict exogeneity.

²See Arellano and Honoré (2001, Chap. 53).

³Notice that this will be zero under the strict exogeneity assumption.

In practice, we base our estimation on the ‘smooth backfitting algorithm’ (see e.g., Mammen et al., 1999 and Nielsen and Sperlich, 2005). For a given j , let us denote $\hat{f}(\mathbf{x}_{it})$ and $\hat{f}(\mathbf{x}_{i,t-1})$ the estimates of $f(\mathbf{x}_{it})$ and $f(\mathbf{x}_{i,t-1})$ respectively. Then, a more precise estimator⁴, say $\hat{\hat{f}}$, can be obtained as a weighted average of $\hat{f}(\mathbf{x}_{it})$ and $\hat{f}(\mathbf{x}_{i,t-1})$:

$$\hat{\hat{f}} = \frac{1}{2} [\hat{f}(\mathbf{x}_{it}) + \hat{f}(\mathbf{x}_{i,t-1})] \quad (4)$$

Below, we apply this methodology to estimate the interplay between the growth rate of GDP per capita and life expectancy at birth.

2.2 Data

We use historical panel data for 18 countries spanning over 1820-2005.⁵ As already mentioned, the variables under investigation are the growth rate of GDP per capita and life expectancy at birth. Data on GDP per capita have been collected from ‘The World Economy: Historical Statistics OECD Development Centre’. We use GDP per capita at 1990 International Geary-Khamis dollars. Life expectancy data are collected from The Human Mortality Database (University of California, Berkeley, and Max Planck Institute for Demographic Research). Life expectancy at birth is the number of years that a newborn baby is expected to live if the age-specific mortality rates effective at the year of birth apply throughout his or her lifetime. Figures 1, 2 and 3 show the evolution of GDP per capita and life expectancy at birth for some countries: North Europe (Sweden, UK), South Europe (France, Spain) and North America (USA, Canada). The pattern does clearly display a co-evolution of the two variables which reflects a stylized fact of economic and demographic transitions: an increase in both GDP per capita and life expectancy at birth.

Insert Figures 1, 2 and 3

We also estimate the density⁶ of the growth rate of GDP per capita and life expectancy at birth for the whole sample (see Figure 4). We observe a unimodal distribution for the growth rate of GDP per capita (about 2.5%) and a bimodal distribution for life expectancy at birth (about 45 years age for the first mode and 74 years age for the second). For the latter, the second mode clearly dominates the first. As a result, we can argue that our sample contains a significant proportion of countries where both low and high values of life expectancy at birth can be observed.

Insert Figure 4

2.3 Estimation results

As mentioned above, our estimates $\hat{f}(\mathbf{x}_{it})$ and $\hat{f}(\mathbf{x}_{i,t-1})$ have closely related shape. We then plot in Figure 5 the weighted average $\hat{\hat{f}}$. With respect to the relational structure, a study of the graph gives the first hand impression that life expectancy effect on per capita income growth rate is

⁴This is particularly useful in case where the shape of the two estimates are closely related.

⁵Australia, Austria, Belgium, Canada, Denmark, Finland, France, Italy, Japan, Netherlands, New Zealand, Norway, Portugal, Spain, Sweden, Switzerland, United Kingdom, USA.

⁶Densities are estimated nonparametrically using the kernel method.

highly non-linear. To test for the significance of non-linearity in the statistical specification, we use the ‘gain’ statistic (see, Hastie and Tibshirani, 1990 for details).⁷ The ‘gain’ is computed as $178.014 > \chi^2(24.008) = 36.424$ at the 5% level. As a result, there is a strong evidence of non-linearity.

Insert Figures 5 and 6

This finding provides a new evidence, in contrast to the linearity assumption of the wide array of empirical models of the demography-economic growth relation built on parametric framework. The curvature suggests that the relation between economic growth and life expectancy involves far more complex mechanism. In the linear case, demographic shocks may eventually wither away with little or no long run effect on economic growth, whereas non-linearity can induce the shocks to work in a much more intricate way.

Our empirical specification is flexible enough to account for the complex way life expectancy does affect economic growth. The main lesson which emerges from Figure 5 is that the relationship between life expectancy and GDP growth rate, while roughly increasing, has quite varying concavity depending on the value of life expectancy. The relationship is for example convex for low life expectancy values, and concave for large enough values of this variable. The finding has been found robust to two modifications. Lagging life expectancy by one period as in Barghava et al (2001) will not affect the convex-concave shape. Moreover, running the same non-parametric estimation on averaged variables over successive 20-years long periods does not smooth out the shape. Hereafter, we study to which extent life-cycle behavior under different demographic structures within an endogenous growth set-up can explain this shape.

3 The benchmark: Blanchard meets Romer

We first very briefly display the basic structure of Blanchard-like models. More details can be found in Blanchard (1985). We depart slightly from the original framework by introducing population growth.⁸ We then introduce learning-by-investing in the spirit of Romer (1986).

3.1 Demography

We assume that at every instant, a cohort is born. Each member of any cohort has a constant instantaneous (flow) probability to die equal to $p > 0$. Therefore, an agent born at μ (generation, cohort or vintage μ) has a probability $e^{-p(z-\mu)}$ to survive at $z > \mu$ and life expectancy is constant and equal to $1/p$. In order to cope with population growth within this vintage structure, we

⁷Intuitively, the ‘gain’ is the difference in normalized deviance between the GAM and the parametric linear model. A large ‘gain’ indicates a lot of non-linearity, at least as regards statistical significance. The distribution of this statistic is approximated by a chi-square $\chi^2(df = df_g - df_l)$, where df_g denotes the degree of freedom of the GAM. It is computed as the trace of $2\mathbf{S} - \mathbf{S}\mathbf{S}'$ where \mathbf{S} is the smoothing matrix, and df_l is the degree of freedom of the parametric linear model. Here we use the first difference linear model $y_{it} - y_{i,t-1} = \beta(\mathbf{x}_{it} - \mathbf{x}_{i,t-1}) + \varepsilon_{it} - \varepsilon_{i,t-1}$, which is then estimated by ordinary least squares. In that case, \mathbf{S} turns out to be the matrix of orthogonal projection: $\mathbf{S} = \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'$, where \mathbf{Z} denotes the matrix of regressors which does stack up elements of $\mathbf{x}_{it} - \mathbf{x}_{i,t-1}$.

⁸A very basic version of Blanchard’s model with population growth can be found in the textbook of Barro and Sala-i-Martin, 1995, pages 110-114.

extend the original Blanchard model (with total population normalized to one at every instant) to have a constant growth rate of population, equal to $n \geq 0$ at any date, say $L(z) = e^{nz}$. Denote by $T(\mu, z)$ the size of a generation born in μ at time $z \geq \mu$. Because of the special mortality assumption assumed just above, we necessarily have: $T(\mu, z) = T(\mu, \mu) e^{-p(z-\mu)}$. Moreover, the initial size of the cohort born at time μ , that is $T(\mu, \mu)$ should be equal to $(p+n) e^{n\mu}$ in order for the size of total population to fulfill

$$\int_{-\infty}^z T(\mu, z) d\mu = \int_{-\infty}^z T(\mu, \mu) e^{-p(z-\mu)} d\mu = e^{nz}.$$

3.2 The consumer's problem

We assume that the utility function is of the standard log form. The intertemporal utility is then:

$$\int_{\mu}^{\infty} \ln C(\mu, z) e^{-(p+\rho)(z-\mu)} dz \quad (5)$$

where $C(\mu, z)$ denotes the consumption of an individual belonging to generation μ at time z , and ρ is the intertemporal discount rate of the utility. Each individual holds an amount of wealth $R(\mu, z)$ which is equal to the accumulated excess of non interest earnings over consumption outlays, plus accumulated interest charges at time z . Agents are constrained to maintain a positive wealth position, and have no bequest motive. Since individual age is directly observed, the annuity rate of interest faced by an individual of age $z-\mu$ is the sum of the world interest rate and his instantaneous probability of dying $(r(z) + p)R(z-\mu)$ at each time, as a payment from the insurance companies if he is still alive. After his death, all his wealth goes to the insurance firms. We consider that all individuals supply entirely their available time, normalized to one, and are paid at the wage rate, $w(z)$, for every date z . Under these considerations the budgetary constraint is:

$$\dot{R}(\mu, z) = (r(z) + p)R(\mu, z) + w(z) - C(\mu, z) \quad (6)$$

with

$$\lim_{z \rightarrow \infty} R(\mu, z) e^{-\int_s^z [r(x)+p]dx} = 0$$

The consumer maximizes Equation (5) subject to (6) with the initial condition $R(\mu, \mu)$ given. We shall assume for simplicity that $R(\mu, \mu) = 0$ for every μ . The associated Hamiltonian is

$$H = \ln C(\mu, z) e^{-(p+\rho)(z-\mu)} + \lambda [(r(z) + p)R(\mu, z) + w(z) - C(\mu, z)]$$

where λ is the co-state variable associated with the state variable $R(\cdot)$. The resulting first order conditions are:

$$\begin{aligned} \frac{\partial H}{\partial C(\mu, z)} &= 0 \Leftrightarrow \frac{1}{C(\mu, z)} e^{-(p+\rho)(z-\mu)} = \lambda \\ \frac{\partial H}{\partial R(\mu, z)} &= -\dot{\lambda} = \lambda [r(z) + p] \Rightarrow \frac{\dot{\lambda}}{\lambda} = -[r(z) + p] \\ \frac{\partial H}{\partial \lambda} &= \dot{R} = [r(z) + p] R(z, \mu) + w(z) - C(z, \mu) \end{aligned}$$

Using the equations above, the optimal consumption is such that:

$$\frac{\partial C(\mu, z)/\partial z}{C(\mu, z)} = \frac{\dot{C}(\mu, z)}{C(\mu, z)} = r(z) - \rho \quad (7)$$

The relation (7) is the traditional Euler equation describing optimal consumption behavior over time. It shows that the optimal path of consumption is closely determined by the difference between the interest rate and the pure rate of time preference. Within Blanchard-Yaari structures, it is then quite straightforward to establish the following proportionality property which will prove key in the famous aggregation properties of the model:

$$C(\mu, z) = (p + \rho)[R(\mu, z) + D(\mu, z)] \quad (8)$$

where

$$D(\mu, z) = \int_z^\infty w(v) e^{-\int_z^v [r(s) + p] ds} dv$$

$D(\mu, z)$ is the human wealth at z of an individual born at μ . It's the present value of the future stream of labor income. $D(\mu, z)$ is obtained by integrating forward the individual's dynamic budget constraint (Equation 6).

Then the aggregation (across surviving cohorts) outcomes are derived neatly, based on the general formula according to which, for any vintage variable $x(\mu, z)$, the aggregate magnitude $X(z)$ is

$$X(z) = \int_{-\infty}^z x(\mu, z) T(\mu, z) d\mu$$

where $T(\mu, z)$ is the size of the cohort μ at time z . In our version with population growth, the resulting law of motions for aggregate consumption, C , aggregate human wealth, D , and aggregate nonhuman wealth, R , are respectively:

$$\dot{C}(z) = (n + r - \rho)C(z) - (p + n)(p + \rho)R(z), \quad (9)$$

$$\dot{D}(z) = [r + p]D(z) - e^{nz} w(z),$$

and

$$\dot{R}(z) = rR(z) + e^{nz} w(z) - C(z).$$

The resulting laws of motion of per capita consumption, c , per capita human wealth, d , and per capita nonhuman wealth, a , are respectively:

$$\dot{c}(z) = (r - \rho)c(z) - (p + n)(p + \rho)a(z), \quad (10)$$

$$\dot{d}(z) = (r + p - n)d(z) - w(z),$$

and

$$\dot{a}(z) = (r - n)a(z) + w(z) - c(z).$$

3.3 The firm's problem

As in Blanchard (1985, p. 232), we consider a closed economy. However we depart from this paper by incorporating an endogenous growth engine, that is a version of the learning-by-doing device of Romer (1986). Firms accumulate capital, and the more they accumulate machines, the more they become expert in them, which boosts their productivity. Productivity growth, and ultimately economic growth, is therefore a side-product of capital accumulation in this model. Notice however that in contrast to Romer's original framework, we have a possibly nonzero population growth in our model. If we incorporate exactly the same externality as Romer (1986), we cannot have balanced growth paths due to the associated scale effect. In this section, we assume that the externality is proportional to the ratio capital to labor, which keeps the initial idea of Arrowian learning-by-doing and solves the scale effect problem. To ease the exposition, we hereby describe briefly the firm model. The production function of a representative firm i is:

$$Y_i = B(K_i)^\varepsilon (AL_i)^{1-\varepsilon}, \quad 0 < \varepsilon < 1$$

where ε denotes the capital share. We suppose that there are N identical and perfectly competitive firms. K_i and L_i are respectively capital and labor factors of firm i . A is labor-saving technical progress. Note that A is not indexed by i , it represents the stock of knowledge of the whole economy, and such a stock is supposed to be outside the control of any particular firm: it is not appropriable. As explained above, we assume that A is an increasing function of the aggregate capital/labor ratio, and to make it even easier, we assume that $A = \frac{K}{L}$, where $K = \sum_{i=1}^N K_i$ and $L = \sum_{i=1}^N L_i$.

As usual, we will work under the assumption of identical firms (symmetric aggregation), which yields $K = N K_i$ and $L = N L_i$. Under zero capital depreciation, the profit function of a representative price-taker firm i is $\pi_i = Y_i - wL_i - rK_i$, and the maximization of this profit with respect to K_i and L_i gives the traditional conditions:

$$\frac{\partial \pi_i}{\partial K_i} = \varepsilon B \left(\frac{K}{N} \right)^{\varepsilon-1} \left(A \frac{L}{N} \right)^{1-\varepsilon} - r = 0 \quad (11)$$

$$\frac{\partial \pi_i}{\partial L_i} = (1-\varepsilon) B \left(\frac{K}{N} \right)^\varepsilon A^{1-\varepsilon} \left(\frac{L}{N} \right)^{-\varepsilon} - w = 0 \quad (12)$$

which, given the externality $A = \frac{K}{L}$, respectively yield $r = \varepsilon B$ and $w = (1-\varepsilon)B k$ where k is the capital/labor ratio.

3.4 General equilibrium

Under the closed economy assumption, nonhuman wealth is equal at equilibrium to aggregate capital stock. In per capita terms, this can be formulated as $a = k$. As usual in one-sector growth models driven by capital accumulation, we are then able to summarize the dynamics of the model at general equilibrium in two equations depending on aggregate consumption and physical capital.⁹ Equation (10) can be rewritten as

$$\dot{c}(z) = (r - \rho)c(z) - (p + n)(p + \rho)k(z), \quad (13)$$

⁹Plus the traditional boundary conditions.

with $r = \varepsilon B$, and the law of motion of k mimics the one of a identified above, and can be rewritten in the following compact form when the optimality conditions $r = \varepsilon B$ and $w = (1 - \varepsilon)B k$ are accounted for:

$$\dot{k}(z) = (B - n) k(z) - c(z) \quad (14)$$

We now study the existence of balanced growth solutions to the system (13)-(14)

3.5 Balanced growth paths and the relationship between longevity and growth

We shall first define steady state growth paths (or balanced growth paths). Since we are dealing with a one-sector model, the balanced growth path concept is simple: we seek to characterize situations in which consumption and capital per capita grow at the same constant rate, say g . Furthermore, as usual in endogenous growth theory, we have indeterminacy in the long-run level of the variables as the two-dimensional system (13)-(14) cannot allow to compute the two long-run levels (of consumption and capital per capita respectively) plus the unknown growth rate g . We shall proceed here by the traditional dimension reduction method. Precisely, we focus on the two variables, ratio consumption to capital, $\frac{c}{k} = X$, and the growth rate $g = \frac{\dot{c}(z)}{c(z)} = \frac{\dot{k}(z)}{k(z)}$, and rewrite the system (13)-(14) at the balanced growth path in terms of these two variables. We get:

$$g = (r - \rho) - (p + n) (p + \rho) X^{-1}, \quad (15)$$

and

$$g = B - n - X. \quad (16)$$

If $p = 0$, we recover the traditional demographic structure in growth theory (with zero capital depreciation), and the counterpart outcomes. Let us depart from this case and assume $p \neq 0$. At first, notice that for the system (15)-(16) to have positive solutions, we must impose that $r > \rho$ and $B > n$, that is the private return to capital accumulation should be greater than the impatience rate, and the productivity parameter B , which plays merely the role of a scale parameter in the final good sector, should be greater than population growth rate. Both conditions are natural and largely acceptable. We also assume $\rho > n$, a very common assumption in growth theory, which will be useful in the proof of main result of this section. Doing the required algebra, it turns out that long-run growth rate g should solve a second-order polynomial as in Aisa and Pueyo (2004):

$$-g^2 + g(B - n + r - \rho) + (B - n) (\rho - r) + (p + n) (p + \rho) = 0 \quad (17)$$

In contrast to Aisa and Pueyo, we are able to analytically characterize the associated properties.

Proposition 1 *Provided $(B - n) (r - \rho) > (p + n) (p + \rho)$, the model displays two strictly positive values for the long-run growth rate g . However, only a single value, the lower, is compatible with a positive ratio consumption to capital.*

Proof. The discriminant of the second-order g-equation (17) is:

$$\Delta = (B - n + r - \rho)^2 + 4((B - n)(\rho - r) + (p + n)(p + \rho))$$

which can be trivially rewritten as:

$$\Delta = (B - n - r + \rho)^2 + 4(p + n)(p + \rho)$$

Therefore, since $p > 0$, we always have two distinct roots. The largest root is necessarily strictly positive once $r > \rho$ and $B > n$ as assumed above. Call it g^M :

$$g^M = \frac{B - n + r - \rho + \sqrt{\Delta}}{2}$$

The second root, say g^m , with $g^m = \frac{B - n + r - \rho - \sqrt{\Delta}}{2}$, has the sign of:

$$(B - n + r - \rho - \sqrt{\Delta})(B - n + r - \rho + \sqrt{\Delta}) = 4((B - n)(r - \rho) - (p + n)(p + \rho))$$

which is positive under $(B - n)(r - \rho) > (p + n)(p + \rho)$. This proves the first part of the proposition.

To prove the second part, notice that g^M is necessarily bigger than B :

$$2g^M = B - n + r - \rho + \sqrt{\Delta} \geq B - n + r - \rho + (B - n - r + \rho) = 2(B - n)$$

since

$$\sqrt{\Delta} = \sqrt{(B - n - r + \rho)^2 + 4(p + n)(p + \rho)} \geq B - n - r + \rho$$

and

$$B - n - r + \rho = (1 - \epsilon)B + \rho - n > 0,$$

since we also assume $\rho > n$. Because the ratio consumption to capital is determined by $X = B - n - g$, X is necessarily negative if $g \geq B - n$. Thus, g^M should be ruled out.

In contrast, by the symmetric argument, g^m checks the inequality $g^m < B - n$. ■

Two comments are in order. First of all, the apparent multiplicity that comes from the second-order polynomial equation is actually fictitious. Therefore, contrary to what is suggested in Aisa and Pueyo (2004), and though we effectively have two distinct and strictly positive values for g , the largest value is simply incompatible with the positivity of the ratio consumption to capital. Second, our sufficient condition is actually not restrictive at all. In particular, if we have in mind that p , n and ρ are generally small numbers, the inequality $(B - n)(r - \rho) > (p + n)(p + \rho)$ or $(B - n)(\epsilon B - \rho) > (p + n)(p + \rho)$ should hold provided the productivity or scale parameter B is not too small. Otherwise, a combination of low productivity (low B) and high mortality (large p) can induce negative growth, which is actually reflected in Figures 5 and 6.¹⁰ In this sense, our model behaves very well. The relationship between growth and longevity induced by the model is even neater as summarized in the following proposition:

¹⁰For brevity, We shall not consider balanced growth paths with negative growth rates. Notice however that given the analytical forms of equilibrium growth rates identified, there is no hope to identify a concavity change in the relationship between growth and life expectancy, even if one allows for negative growth rates. The original Blanchard-Yaari structures are definitely not suitable for that.

Proposition 2 *Under the assumptions of Proposition 1, the unique admissible long-run growth rate of the economy is a strictly increasing, strictly concave function of longevity. In other words, g^m is a strictly decreasing, strictly convex function of p .*

The proof is trivial by simple differentiation of $g^m = \frac{B-n+r-\rho-\sqrt{\Delta}}{2}$ and its first derivative with respect to p , given $\Delta = (B - n - r + \rho)^2 + 4(p + n)(p + \rho)$. The main conclusion of this section is therefore that the benchmark model obtained by combination of the perpetual youth model of Blanchard (1985) and the learning-by-investing engine of Romer (1986) delivers a quite simple picture of the relationship between longevity and growth: the relationship is strictly monotonic and strictly concave, and it does not exhibit any first-order difference between the case of low and high life expectancy countries. In the following sections, we examine some extensions of the model seen above departing from the spirit of the “perpetual youth” setting, first allowing for age-dependent labor income, then accounting for age-dependent survival probabilities. As we have shown that population growth, that is parameter n , has no crucial role in shaping the relationship between growth and life expectancy, we zero population growth ($n = 0$) hereafter to ease the algebra, and we normalize total population to 1 so that aggregate and per-capita variables coincide.

4 A model with a more realistic age-profile of earnings

In the previous model, we implicitly assume that any individual in the economy will receive a wage $w(z)$ at time z irrespective of its age, which is clearly unrealistic. In this section, we relax this assumption and consider age-dependent labor income.

4.1 The modified model

We shall extend the benchmark model by considering the following change in the production function:

$$Y_i = B(K_i)^\varepsilon (A \tilde{L}_i)^{1-\varepsilon}, \quad 0 < \varepsilon < 1$$

where \tilde{L}_i is given by at any time z by

$$\tilde{L}_i(z) = \int_{-\infty}^z \Omega(z - \mu) L_i(\mu, z) d\mu,$$

and

$$\Omega(z - \mu) = a_1 e^{-\alpha_1(z-\mu)} + a_2 e^{-\alpha_2(z-\mu)},$$

being a_1 , α_1 and α_2 positive parameters and a_2 negative. \tilde{L}_i measures therefore a kind of age-adjusted effective labor. The age weighting function $\Omega(z - \mu)$ is taken from Faruquee et al. (1997). The first exponential term in $\Omega(z - \mu)$ captures the gradual decline in labor due to aging and the second one goes in the other direction and captures the gains from experience (and therefore from longer lives). Accordingly, $L_i(\mu, z)$ measures the size of workers of age $z - \mu$ at time z .

A few comments are in order here regarding the interpretation of the model, and notably the age weights, $\Omega(z - \mu)$. As in Faruquee et al. (1997), we have to make sure that $\Omega(z - \mu)$ starts

increasing with age then decreases after a while once the gains from experience get dominated by labor decline due to aging. This is of course ensured by an adequate choice of the parameters a_i and α_i , $i = 1, 2$, and we will come back to this point later. Let us focus at the minute on a more immediate matter: As it stands, and provided the weight function $\Omega(z - \mu)$ is positive as it should be, we are formalizing a case when except when $z = \mu$, that it is roughly speaking, except when the age of the worker is strictly zero, all individuals of all cohorts are working, even those whose age $z - \mu$ tends to zero. Naturally, we could have altered the model by assuming that individuals of any vintage μ only work from $\mu + T_0$, with $T_0 > 0$ a given positive number, and then adjust the other specifications to this modification. We have preferred to keep a modelling as close as possible to the benchmark model, and not introduce the latter modification. In the interpretation of the numerical results below, one should however interpret the age zero here, not as a biological age, but as the start of the planning period of the individuals.

As mentioned in the end of the previous section, we shall simplify from now the algebra (and without any loss of generality) by assuming $n = 0$ and coming back to the original Romer's externality formulation, $A = K$. In such a case, the size of population with age $z - \mu$ at date z is simply $p e^{-p(z-\mu)}$, and firms profit-maximization under symmetric aggregation yields in particular:

$$w(\mu, z) = B(1 - \epsilon) \tilde{L}^{-\epsilon}(z) \Omega(z - \mu) K(z),$$

with

$$\tilde{L}(z) = \int_{-\infty}^z \Omega(z - \mu) p e^{-p(z-\mu)} d\mu,$$

where $w(\mu, z)$ is the wage paid to people of age $z - \mu$ at date z . The integral just above can be computed exactly and it turns out to be a constant. For example, if $a_1 = -a_2 = a > 0$, we get:

$$\tilde{L}(z) = \tilde{L} = a p \left[\frac{1}{\alpha_1 + p} - \frac{1}{\alpha_2 + p} \right].$$

We need $\alpha_1 < \alpha_2$ for $\tilde{L} > 0$, which is again consistent with Faruquee et al. (1997) and also with Japelli and Pagano (1989). Notice that the latter condition (coupled with $a_1 = -a_2$) also ensures that the age weight function $\Omega(z - \mu)$ is positive and starts increasing with age (hump-shaped). The wage per age $w(\mu, z)$ can be then written as:

$$w(\mu, z) = \Omega(z - \mu) \tilde{w}(z), \tag{18}$$

where

$$\tilde{w}(z) = B(1 - \epsilon) \tilde{L}^{-\epsilon} K(z),$$

plays the role of age-unadjusted wage. Notice that in our modified framework, firms maximization with respect to capital yields the following equation for the interest rate

$$r = \epsilon B \tilde{L}^{1-\epsilon},$$

which therefore remains constant at equilibrium as before.

We now proceed to the consumer side and specially to the counterpart of the aggregation formulas. We shall in particular examine whether aggregation “kills” the nonlinearity gathered by the age-profile of earnings as featured in equation (18). Let us first start mentioning that the consumer problem is unchanged except that the budgetary constraint should now incorporate the age-dependent feature of life earnings:

$$\dot{R}(\mu, z) = (r(z) + p)R(\mu, z) + w(\mu, z) - C(\mu, z) \quad (19)$$

As an implication of this unique difference, we get the same proportionality property as in the benchmark model but with a different definition for human wealth, that is

$$C(\mu, z) = (p + \rho)[R(\mu, z) + D(\mu, z)],$$

where

$$D(\mu, z) = \int_z^\infty w(\mu, v) e^{-\int_z^v [r(s) + p] ds} dv.$$

The following proposition gives the corresponding law of motion for aggregate human wealth.

Proposition 3 *The aggregate human wealth, $D(z) = \int_{-\infty}^z D(\mu, z) p e^{-p(z-\mu)} d\mu$, follows the law of motion:*

$$\dot{D}(z) = (r + p) D(z) + \int_z^\infty \frac{\partial \psi(v, z)}{\partial z} \tilde{w}(v) e^{-(r+p)(v-z)} dv - \tilde{L} \tilde{w}(z),$$

where

$$\psi(v, z) = \int_{-\infty}^z \Omega(\mu, v) p e^{-p(z-\mu)} d\mu.$$

Proof. Notice that by definition of the terms $D(\mu, z)$, we have:

$$D(z) = \int_{-\infty}^z \int_z^\infty w(\mu, v) e^{-\int_z^v [r(s) + p] ds} dv p e^{-p(z-\mu)} d\mu,$$

and by permuting the integration order and using the wage equation (18),

$$D(z) = \int_z^\infty \left[\int_{-\infty}^z \Omega(\mu, v) p e^{-p(z-\mu)} d\mu \right] \tilde{w}(v) e^{-\int_z^v [r(s) + p] ds} dv.$$

We can now differentiate the previous equation with respect to z , and get:

$$\dot{D}(z) = -\psi(z, z) \tilde{w}(z) + \int_z^\infty \frac{\partial \psi(v, z)}{\partial z} \tilde{w}(v) e^{-\int_z^v [r(s) + p] ds} dv + (r(z) + p) D(z),$$

where

$$\psi(v, z) = \int_{-\infty}^z \Omega(\mu, v) p e^{-p(z-\mu)} d\mu.$$

One can trivially perform the integration just above and obtain:

$$\psi(v, z) = a p \left[\frac{e^{-\alpha_1(v-z)}}{\alpha_1 + p} - \frac{e^{-\alpha_2(v-z)}}{\alpha_2 + p} \right],$$

so that

$$\psi(z, z) = a p \left[\frac{1}{\alpha_1 + p} - \frac{1}{\alpha_2 + p} \right] = \tilde{L}.$$

Using finally the constancy of the interest rate at equilibrium ends the proof. \blacksquare

It is worth pointing out that with respect to the benchmark model, the law of motion of aggregate human capital is significantly altered due to the new integral term $\int_z^\infty \frac{\partial \psi(v, z)}{\partial z} \tilde{w}(v) e^{-(r+p)(v-z)} dv$, which captures the impact of the age-dependency of earning profiles on the dynamics of human wealth. This age-dependency factor is also shown to arise in the law of motion of aggregate consumption, as reflected in the next proposition.

Proposition 4 *The aggregate nonhuman wealth, $R(z) = \int_{-\infty}^z R(\mu, z) p e^{-p(z-\mu)} d\mu$, follows the law of motion:*

$$\dot{R}(z) = rR(z) + \tilde{L} \tilde{w}(z) - C(z).$$

Aggregate consumption follows:

$$\dot{C}(z) = (r - \rho)C - p(p + \rho)R(z) + (p + \rho) \int_z^\infty \frac{\partial \psi(v, z)}{\partial z} \tilde{w}(v) e^{-(r+p)(v-z)} dv.$$

Proof. By differentiation of the definition of $R(z)$ with respect to z , one gets:

$$\dot{R}(z) = -pR(z) + \int_{-\infty}^z \dot{R}(\mu, z) p e^{-p(z-\mu)} d\mu.$$

By the budget constraint (19), one gets immediately after recognizing aggregate consumption:

$$\dot{R}(z) = -pR(z) - C(z) + \int_{-\infty}^z w(\mu, z) p e^{-p(z-\mu)} d\mu.$$

But:

$$\int_{-\infty}^z w(\mu, z) p e^{-p(z-\mu)} d\mu = \int_{-\infty}^z \Omega(\mu, z) \tilde{w}(z) p e^{-p(z-\mu)} d\mu,$$

and

$$\int_{-\infty}^z \Omega(\mu, z) p e^{-p(z-\mu)} d\mu = \tilde{L},$$

we get therefore the law of motion of nonhuman wealth stated in the proposition. The law of motion of aggregate consumption is then trivial using the proportionality $C(z) = (p + \rho) (R(z) + D(z))$ and the laws of motion for both aggregate human and nonhuman wealth already identified above. \blacksquare

We can now study the steady state properties of the general equilibrium outcome of this model.

4.2 Balanced growth paths and the relationship between growth and longevity

We now mimic the job done in Sections 3.4 and 3.5 on the benchmark model. As the economy is still closed, we replace aggregate nonhuman wealth, $R(z)$ by the stock of capital $K(z)$ in the law of motion of the former given in the previous proposition and get:

$$\dot{K}(z) = rK(z) + \tilde{L} \tilde{w}(z) - C(z),$$

which yields using the equilibrium expressions for r and $\tilde{w}(z)$:

$$\dot{K}(z) = B \tilde{L}^{1-\epsilon} - C(z).$$

Along the balanced growth paths, we get the same kind of equation as in the benchmark model, that is

$$g = \tilde{B} - X,$$

with $\tilde{B} = B \tilde{L}^{1-\epsilon}$. The second equation in g and X is obtained as before from the law of motion of aggregate consumption once $R(z)$ and $\tilde{w}(z)$ are replaced by their equilibrium expressions in $K(z)$:

$$\dot{C}(z) = (r - \rho)C - p(p + \rho)K(z) + B(1 - \epsilon)\tilde{L}^{-\epsilon} (p + \rho) \int_z^\infty \frac{\partial \psi(v, z)}{\partial z} K(v) e^{-(r+p)(v-z)} dv.$$

Written along the balanced growth paths, this equation produces a well-defined g-equation once the ratio consumption to capital, X , is replaced by $X = \tilde{B} - g$:

$$g = (r - \rho) - \frac{p(p + \rho)}{\tilde{B} - g} \left[1 - B(1 - \epsilon)\tilde{L}^{-\epsilon} (p + \rho) \int_z^\infty \frac{\partial \psi(v, z)}{\partial z} e^{-(r+p-g)(v-z)} dv \right], \quad (20)$$

with

$$\psi(v, z) = a p \left[\frac{e^{-\alpha_1(v-z)}}{\alpha_1 + p} - \frac{e^{-\alpha_2(v-z)}}{\alpha_2 + p} \right].$$

After noticing that

$$\int_z^\infty \frac{\partial \psi(v, z)}{\partial z} e^{-(r+p-g)(v-z)} dv = \frac{ap\alpha_1}{(\alpha_1 + p)(r + p + \alpha_1 - g)} - \frac{ap\alpha_2}{(\alpha_2 + p)(r + p + \alpha_2 - g)},$$

one can figure out that the g-equation obtained is equivalent to a fourth degree polynomial, and therefore definitely much less analytically tractable than the corresponding one in the benchmark case. Indeed, the balanced growth paths' growth rate, g , solves the equation:

$$F(g) = g - (r - \rho) + \frac{p(p + \rho)}{\tilde{B} - g} \left[1 - B(1 - \epsilon)\tilde{L}^{-\epsilon} (p + \rho) \left(\frac{\Omega_1}{r + p + \alpha_1 - g} - \frac{\Omega_2}{r + p + \alpha_2 - g} \right) \right] = 0, \quad (21)$$

with $\Omega_i = \frac{ap\alpha_i}{\alpha_i + p}$, $i = 1, 2$. While it is impossible to characterize as completely as in Section 3 the solutions of this equation, it is possible to set an existence result of regular positive solutions in g and X , as we will see in a minute. However, there is no hope to have even partial analytical results on the relationship between the eventual long-run growth rate g and life expectancy, say $q = \frac{1}{p}$, given that the following parameters involved in the equation above, that is: Ω_1 , Ω_2 , \tilde{B} and r (since $r = \epsilon B \tilde{L}^{1-\epsilon}$, and \tilde{L} depends on of p), are all non-trivial functions of p . As in related papers (see Faruquee et al., 1997), we shall resort to numerical simulations to investigate the latter issue. Before, let us state the following existence proposition.

Proposition 5 *If the productivity parameter B is large enough, and the difference $\alpha_2 - \alpha_1$ is a small enough positive number, then there exists a positive growth rate g solving equation (21) such that the corresponding consumption to capital ratio is also positive.*

Proof. For $X > 0$, one should ensure that $g < \tilde{B}$ since $X = \tilde{B} - g$. So we study the equation (21) in the interval $[0, \tilde{B}]$. Notice that

$$F(0) = -(r - \rho) + \frac{p(p + \rho)}{\tilde{B}} \left[1 - B(1 - \epsilon)\tilde{L}^{-\epsilon}(p + \rho) \left(\frac{\Omega_1}{r + p + \alpha_1} - \frac{\Omega_2}{r + p + \alpha_2} \right) \right],$$

so that $F(0)$ has the sign of $\rho - r$ under the two assumptions of the proposition (B large enough and $\alpha_2 - \alpha_1$ small enough). If $r > \rho$ applies, which is the case here as in the benchmark model, then $F(0) < 0$. Similarly, the two assumptions guarantee that $F(g)$ tends to $+\infty$ when g goes to \tilde{B} , which ends the proof. ■

We now move to the numerical study of the relationship between g and $q = \frac{1}{p}$. As in Faruquee et al. (1997), we fix $\alpha_1 = 0.06$ and $\alpha_2 = 0.1$.¹¹ We vary q from 10 to 60 and study the corresponding g -solution to (21). We check uniqueness of the balanced growth paths' growth rate (with positive consumption to capital ratio, that is with $g < \tilde{B}$) at every step. The results are reported in Figure 7 below.

Insert Figure 7

Mimicking the shape of the age weight function $\Omega(z - \mu)$, the relationship between economic growth and life expectancy is hump-shaped: at lower values of life expectancy, further longevity gains boost growth but in aging population, such a move increases the proportion of aged people above the age threshold from which income goes down, so that eventually the growth rate of the economy ends up falling.¹² We get the typical story told in Kelley and Schmidt (1995) and theorized in Boucekkine et al. (2002) in a much more sophisticated vintage human capital model. But we cannot find any convexity at low or very low values of life expectancy. The whole (natural) story here is that the obtained shape for the relationship between life expectancy and growth does mimic the hump-shaped nature of the age profile $\Omega(z - \mu)$. Hereafter, we show that age-dependent survival probabilities can instead generate the convex-concave feature we are seeking to reproduce.

5 A model with more realistic demography

Rather than a typical Blanchard-like set-up, we choose the survival law previously put forward by Boucekkine et al. (2002). The probability of surviving until age a ($a = z - \mu$) for any individual of cohort μ is

$$m(a, \mu) = \frac{e^{-\beta(\mu)a} - \alpha(\mu)}{1 - \alpha(\mu)} \quad (22)$$

and the probability of death at age a is

$$F(a, \mu) = 1 - \frac{e^{-\beta(\mu)a} - \alpha(\mu)}{1 - \alpha(\mu)} = 1 - m[\alpha(\mu), \beta(\mu), (z - \mu)] = \frac{1 - e^{-\beta(\mu)a}}{1 - \alpha(\mu)} \quad (23)$$

¹¹The other parameters are fixed as follows: $a = 1$, $B = 0.75$, $\epsilon = 0.3$ and $\rho = 0.01$. Needless to say, we test the robustness of our results to changes in the parameterization, precisely with respect to a and B 's values.

¹²Again, as mentioned above, the figure obtained should be interpreted having in mind that age 0 is the age at which individuals start working and/or planning.

with $\beta(\mu)$ an indicator of survival for old persons, and $\alpha(\mu)$ is an indicator of survival for young persons. We suppose $\beta(\mu) < 0$, and $\alpha(\mu) > 1$ as in Boucekkine et al. (2002) in order to generate a concave survival law as observed in real life, as described in Figure 8.

Insert Figure 8

The maximum age possible for individuals of cohort μ is given by: $m(a, \mu) = 0$, that it is, $A_{\max} = -\frac{\ln(\alpha(\mu))}{\beta(\mu)}$. The expression of the instantaneous probability of dying is then:

$$S(a) = \frac{\partial F(a)/\partial z}{m} = \frac{-\partial m/\partial z}{m} = \frac{\beta(\mu)e^{-\beta(\mu)(z-\mu)}}{e^{-\beta(\mu)(z-\mu)} - \alpha(\mu)} \quad (24)$$

Life expectancy is:

$$E = \int_{\mu}^{\infty} (z - \mu) \frac{\beta(\mu)e^{-\beta(\mu)(z-\mu)}}{1 - \alpha(\mu)} dz = \frac{\alpha(\mu) \ln \alpha(\mu)}{\beta(\mu)(1 - \alpha(\mu))} + \frac{1}{\beta(\mu)} \quad (25)$$

For $\beta(\mu) > 0$ and $\alpha(\mu) \rightarrow 0$, one finds the Blanchard result. The size of the population at time z is

$$T_z = \int_{z-A_{\max}}^z \xi e^{n\mu} \frac{e^{\beta(\mu)(z-\mu)} - \alpha(\mu)}{1 - \alpha(\mu)} d\mu \quad (26)$$

5.1 The model

With respect to the benchmark model, we keep the production side unchanged with the exception of population growth which we zero here, total population size being normalized to 1 as in Blanchard (1985). Nonetheless, we modify substantially the consumer side in order to incorporate more realistic demographics. We therefore concentrate on the latter problem hereafter. We shall consider the optimization problem of an individual of a generation μ . For ease of exposition we may omit the dependence of the demographic parameters $\alpha(\mu)$ and $\beta(\mu)$ on μ . Assuming that the instantaneous utility derived from consumption is logarithmic for an individual born in μ still living in z , the intertemporal utility is

$$\int_{\mu}^{\mu+A_{\max}} \ln C(\mu, z) \frac{e^{-\beta(z-\mu)} - \alpha}{1 - \alpha} e^{-\rho(z-\mu)} dz \quad (27)$$

One can consider that utility after the individual dies is equal to zero, then in the intertemporal utility we can replace $\mu + A_{\max}$ by ∞ . On the other hand, and contrary to Boucekkine et al. (2002), we assume that there is no disutility of work. As in the precedent section we find the traditional Euler equation

$$\frac{\partial C(\mu, z)/\partial z}{C(\mu, z)} = \frac{\dot{C}}{C} = r(z) - \rho + S(\mu, z) - S(\mu, z) = r(z) - \rho \quad (28)$$

That is to say $C(\mu, z) = C(\mu, \mu) e^{\int_{\mu}^z (r(s) - \rho) ds}$. Consumption over time can be characterized much more finely using the approach highlighted in Faruquee (2003).

Proposition 6

$$C(\mu, z) = \phi(\mu, z)[R(\mu, z) + D(\mu, z)] \quad (29)$$

where

$$\phi(\mu, z) = \frac{1}{\int_z^\infty e^{-\int_z^v [\rho + S(\alpha, \beta, (x-\mu))] dx} dv}$$

Proof. See appendix. ■

Corollary 1 *If $S(\mu, z) = p$, then Equation (29) degenerates into the Blanchard's case: $C(\mu, z) = (p + \rho)[R(\mu, z) + D(\mu, z)]$.*

$\phi(\mu, z)$ is the marginal propensity to consume. Contrary to the Blanchard case, the marginal propensity to consume is no longer constant, it is a much more complicated and depends in particular on age and generation characteristics:

$$\phi(\mu, z) = \frac{\rho(\rho + \beta)(e^{-\beta a} - \alpha)}{\rho [e^{-\beta a} - e^{(\rho + \beta)(a - A_{\max})}] + \alpha(\rho + \beta) [e^{\rho(a - A_{\max})} - 1]} \quad (30)$$

and

$$\phi(\mu, \mu) = \frac{\rho(\rho + \beta)(1 - \alpha)}{\rho [1 - e^{-A_{\max}(\rho + \beta)}] + \alpha(\rho + \beta) [e^{-\rho A_{\max}} - 1]} \quad (31)$$

One can then notice that contrary to the Blanchard case previously studied by Aisa and Pueyo (2004), the marginal propensity to consume, in addition to be age-dependent, is a definitely much more complex function of the demographic and preference parameters. In order to get a closer idea about this, let us study the evolution of $\phi(\mu, \mu)$ with respect to β , α , and ρ . We first give the exact algebraic expressions of the derivatives involved before stating a proposition and providing some numerical illustrations.

$$\begin{aligned} \frac{\partial \phi(\mu, \mu)}{\partial \beta} &= \frac{e^{-(\rho + \beta)A_{\max}} \left[(1 - \alpha) \left(-\rho^2 + \frac{\rho^3(\rho + \beta)}{\beta} \right) \right] - \frac{e^{-\rho A_{\max}} \rho^2 \alpha (1 - \alpha) (\rho + \beta)^2}{\beta} + \rho^2 (1 - \alpha)}{[\rho (1 - e^{-(\rho + \beta)A_{\max}}) + \alpha(\rho + \beta)(e^{-\rho A_{\max}} - 1)]^2} \\ \frac{\partial \phi(\mu, \mu)}{\partial \alpha} &= \frac{e^{-(\rho + \beta)A_{\max}} \rho^2 (\rho + \beta) \left[\frac{(1 - \alpha)\rho + \beta}{\alpha \beta} \right] - e^{-\rho A_{\max}} \rho (\rho + \beta)^2 \left[1 + \frac{\rho(1 - \alpha)}{\beta} \right] + \beta \rho (\rho + \beta)}{[\rho (1 - e^{-(\rho + \beta)A_{\max}}) + \alpha(\rho + \beta)(e^{-\rho A_{\max}} - 1)]^2} \\ \frac{\partial \phi(\mu, \mu)}{\partial \rho} &= \frac{-e^{-(\rho + \beta)A_{\max}} \rho^2 (1 - \alpha) [(\rho + \beta)A_{\max} + 1] + e^{-\rho A_{\max}} \alpha (1 - \alpha) (\rho + \beta)^2 [\rho A_{\max} + 1]}{[\rho (1 - e^{-(\rho + \beta)A_{\max}}) + \alpha(\rho + \beta)(e^{-\rho A_{\max}} - 1)]^2} \\ &\quad + \frac{(1 - \alpha) [-\alpha (\rho + \beta)^2 + \rho^2]}{[\rho (1 - e^{-(\rho + \beta)A_{\max}}) + \alpha(\rho + \beta)(e^{-\rho A_{\max}} - 1)]^2} \end{aligned}$$

We can then state the following property:

Proposition 7 *If the maximum age A_{\max} is large enough (if α large enough and/or β close to zero), then $\frac{\partial \phi(\mu, \mu)}{\partial \alpha} < 0$, $\frac{\partial \phi(\mu, \mu)}{\partial \beta} < 0$ and $\frac{\partial \phi(\mu, \mu)}{\partial \rho} > 0$*

Proof. See appendix. ■

5.2 Numerical exercises

Proposition 7 is illustrated in Figures 9, 10 and 11, where the saving rate, $s(\mu, \mu)$, which is equal to $1 - \phi(\mu, \mu)$, is represented as a function of the three parameters considered. Economically speaking, the derivatives are correctly signed in Proposition 7. If demographic conditions move in such a way that life expectancy and/or the maximal age go up (α and β increasing) then the consumer will face higher horizons and save more in marginal terms. In contrast, when the impatience rate is raised (ρ growing), the propensity to consume increases, and the saving rate goes down. This is clearly reflected in Figures 9 to 11 for some reasonable parameterizations of the model.

Insert Figures 9, 10 and 11

In order to study whether the properties outlined just above are truly sensitive to the sufficient condition of Proposition 7, that is to a sufficiently large value of the maximal age, we have run more numerical experiments. A sample is given in Figures 12, 13 and 14 in which the range of values taken by the maximal age, A_{\max} , is much tighter than in the first case. Again, we recover the same patterns, suggesting that the properties are indeed much less fragile than what could be inferred from the statement of Proposition 7.

Insert Figures 12, 13 and 14

We end this section by considering a very important property of the model already mentioned in the introduction. In contrast to Aisa and Pueyo (2004), the saving rates or propensities to save do depend on the age of the individuals. Intuitively, the older should have the lower propensities to save. Figure 15 shows an illustration of this property of the model: as one can see, the evolution of the saving rate for an individual born at μ still living at z is clearly declining: an individual saves definitely much less when old compared to her youth. This should induce some strong non-linearity between longevity and development. If the latter relies on accumulation of physical capital, and if such an accumulation is only possible thanks to domestic savings, then a larger longevity has also a negative effect on growth by increasing the proportion of people with relatively small saving rates. Of course it is not clear at all whether this negative effect will dominate the direct positive effects of increasing longevity, but we can already argue that the relationship between the latter and economic development cannot be as simple as in the typical Blanchard-like models with physical capital accumulation.

Insert Figure 15

5.3 Aggregation

Before studying the growth rate of the aggregate economy, we need to define and compute some aggregate figures across generations or vintages. Given the characteristics of our model, for any vintage $x_{\mu, z}$, the aggregate magnitude $X(z)$ is computed following:

$$X(z) = \int_{-\infty}^z x(\mu, z)T(\mu, z)d\mu = \int_{z-A_{\max}}^z x(\mu, z)T(\mu, z)d\mu$$

where $T(\mu, z)$ is the size of generation μ . We start with aggregate consumption.

Proposition 8 *The aggregate consumption $C(z) = \int_{z-A_{\max}}^z C(\mu, z)T(\mu, z)d\mu$ evolves according to*

$$\dot{C}(z) = \xi\phi(z, z)D(z, z) + (r(z) - \rho)C(z) - \int_{z-A_{\max}}^z C(\mu, z)S(\mu, z)T(\mu, z)d\mu \quad (32)$$

with $T(\mu, z) = \xi m[\beta, \alpha, (z - \mu)]$ and $C(\mu, z) = C(\mu, \mu)e^{\int_{\mu}^z [r(s) - \rho]ds}$.

Proof. From Equation (28) we can deduce

$$C(\mu, z) = C(\mu, \mu)e^{\int_{\mu}^z [r(s) - \rho]ds} \quad (33)$$

Aggregate consumption is

$$C(z) = \int_{-\infty}^z C(\mu, z)T(\mu, z)d\mu$$

Differentiating the latter equation with respect to z , one gets:

$$\dot{C} = C(z, z)T(z, z) + \int_{-\infty}^z \dot{C}(\mu, z)T(\mu, z)d\mu + \int_{-\infty}^z C(\mu, z)\dot{T}(\mu, z)d\mu$$

Using again Equation (28) we can go further:

$$\begin{aligned} \dot{C} &= C(z, z)T(z, z) + \int_{-\infty}^z (r(z) - \rho)C(\mu, z)T(\mu, z)d\mu + \int_{-\infty}^z C(\mu, z)\dot{T}(\mu, z)d\mu \\ &= C(z, z)T(z, z) + (r(z) - \rho)C(z) + \int_{-\infty}^z C(\mu, z)\dot{T}(\mu, z)d\mu \end{aligned}$$

We know that $T(\mu, z) = \xi m(z - \mu)$, which implies that $\dot{T}(\mu, z) = \xi \dot{m}(z - \mu)$. Since $S(z - \mu) = -\frac{\dot{m}(z - \mu)}{m}$, we deduce that $\dot{m}(z - \mu)$ is equal to $-m(z - \mu)S(z - \mu)$, and $\dot{T}(\mu, z)$ is equal to $-\xi m(z - \mu)S(z - \mu)$. As a result, the law of motion of aggregate consumption can be rewritten again as:

$$\dot{C} = C(z, z)T(z, z) + (r(z) - \rho)C(z) - \int_{-\infty}^z C(\mu, z)S(\mu, z)T(\mu, z)d\mu$$

$C(z, z)$ is the consumption of newly born at date z . We know by Proposition 6 that $C(z, z) = \phi(z, z)D(z, z)$ with $R(z, z) = 0$ since we assumed that a newly born agent has no financial wealth. We also have $T(z, z) = \xi m(z, z) = \xi \frac{e^{-\beta(z-z)} - \alpha}{1 - \alpha}$, implying that $T(z, z) = \xi$. This yields the law of motion stated in this proposition:

$$\dot{C} = \xi\phi(z, z)D(z, z) + (r(z) - \rho)C(z) - \int_{-\infty}^z C(\mu, z)S(\mu, z)T(\mu, z)d\mu \quad \blacksquare$$

This law of motion is markedly different from Blanchard's (1985) as re-used by Aisa and Pueyo (2004). In our model, aggregate consumption depends on three terms: the human wealth of newly born, the Keynes-Ramsey standard term (the difference between the interest rate and the pure rate of time preference), and the expected consumption forgone by individuals dying at z . In particular this last term does not show up in the typical Blanchard aggregation formulas. This term makes a big difference and complicates substantially the computations with respect to standard Blanchard model. The next two propositions states two useful aggregation formulas for human and non-human wealth respectively, which again show substantial departures from the standard case. The evolution of human wealth $D(z)$ below stated.

Proposition 9 *The total human wealth at z , $D(z) = \int_{z-A_{\max}}^z D(\mu, z)T(\mu, z)d\mu$ evolves according to:*

$$\begin{aligned}\dot{D}(z) &= \xi D(z, z) - w(z)L(z) \\ &\quad + \int_{z-A_{\max}}^z \int_z^{z+A_{\max}} w(v)[r(z) + S(z-\mu)]e^{-\int_z^v [r(s)+S(s-\mu)]ds} T(\mu, z) dv d\mu \\ &\quad - \int_{z-A_{\max}}^z D(\mu, z)T(\mu, z)S(z-\mu)d\mu\end{aligned}\tag{34}$$

Proof. See appendix. ■

For the non-human wealth variable, $R(z)$, we have the following aggregation formula:

Proposition 10 *The evolution of the aggregate nonhuman wealth $R(z) = \int_{z-A_{\max}}^z R(\mu, z)T(\mu, z)d\mu$ is*

$$\dot{R}(z) = R(z, z)T(z, z) + r(z)R(z) + w(z)L(z) - C(z)$$

Proof. See appendix. ■

We now turn to the computation of the steady state growth rates and their relationship with the demographic parameters.

5.4 The balanced growth paths

Integrating the Euler equation (28) (with $R(\mu, \mu) = 0$), then replacing the obtained formula for $C(\mu, z)$ in the definition of aggregate consumption, one gets once Proposition 6 is used for an explicit representation of $C(\mu, \mu)$:

$$\begin{aligned}C(z) &= \int_{-\infty}^z \phi(\mu, \mu)D(\mu, \mu)e^{\int_{\mu}^z [r-\rho]ds} T(\mu, z)d\mu \\ &= \int_{z-A_{\max}}^z \phi(\mu, \mu)D(\mu, \mu)e^{\int_{\mu}^z [r-\rho]ds} T(\mu, z)d\mu\end{aligned}$$

where

$$D(\mu, \mu) = \int_{\mu}^{\mu+A_{\max}} w(v)e^{-\int_{\mu}^v [r(s)+S(s-\mu)]ds} dv$$

and

$$w(z) = (1 - \varepsilon)BKL^{-\varepsilon} = \bar{G}K$$

with $\bar{G} = (1 - \varepsilon)BL^{-\varepsilon}$. In order to have an explicit characterization of $C(z)$, we therefore need explicit forms for $L(z)$ and $D(\mu, \mu)$. This is done hereafter. First note that because we assume zero demographic growth in our model, the labor force is constant and equal to:

$$L = \int_{z-A_{\max}}^z \xi e^{n\mu} \frac{e^{\beta(\mu, z-\mu)} - \alpha(\mu)}{1 - \alpha(\mu)} d\mu = \frac{\xi}{1 - \alpha} \left[\frac{1 - e^{-\beta A_{\max}}}{\beta} - \alpha A_{\max} \right]$$

As to $D(\mu, \mu)$, we can easily refine its expression along the steady state. Since we are looking for exponential solutions for K at rate g , say $K(z) = \bar{K} e^{gz}$, with \bar{K} a constant, we obtain:

$$\begin{aligned}
D(\mu, \mu) &= \bar{G} \bar{K} \int_{\mu}^{\mu+A_{\max}} e^{gv} e^{-\int_{\mu}^v [r+S(s-\mu)] ds} dv \\
&= \bar{G} \bar{K} \int_{\mu}^{\mu+A_{\max}} e^{gv} e^{-r(v-\mu) + \ln\left(\frac{e^{-\beta(v-\mu)} - \alpha}{e^{-\beta(\mu-\mu)} - \alpha}\right)} dv \\
&= \bar{G} \bar{K} \int_{\mu}^{\mu+A_{\max}} e^{gv} e^{-r(v-\mu)} \frac{e^{-\beta(v-\mu)} - \alpha}{e^{-\beta(\mu-\mu)} - \alpha} dv \\
&= \frac{\bar{G} \bar{K} e^{g\mu}}{1-\alpha} \left[\frac{e^{A_{\max}(g-r-\beta)} - 1}{g-r-\beta} - \frac{\alpha(e^{A_{\max}(g-r)} - 1)}{g-r} \right] \\
&= \bar{K} \bar{P}(g) e^{g\mu}
\end{aligned}$$

with $\bar{P}(g) = \frac{\bar{G}}{1-\alpha} \left[\frac{e^{A_{\max}(g-r-\beta)} - 1}{g-r-\beta} - \frac{\alpha(e^{A_{\max}(g-r)} - 1)}{g-r} \right]$. Then aggregate consumption can be much more finely characterized as:

$$\begin{aligned}
C(z) &= \xi \phi(\mu, \mu) \bar{K} \bar{P}(g) \int_{z-A_{\max}}^z e^{g\mu} e^{(r-\rho)(z-\mu)} \frac{e^{-\beta(z-\mu)} - \alpha}{1-\alpha} d\mu \\
C(z) &= \frac{\xi \phi(\mu, \mu) \bar{K} \bar{P}(g) e^{gz}}{1-\alpha} \left[\frac{1 - e^{A_{\max}(r-\beta-\rho-g)}}{g-r+\beta+\rho} - \frac{\alpha(1 - e^{A_{\max}(r-\rho-g)})}{g-r+\rho} \right] \quad (35)
\end{aligned}$$

Now let use that since we are along a balanced growth path, we can parameterize $C(z)$ as follows $C(z) = \bar{C} e^{gz}$, with \bar{C} a constant. As before, we denote by $\frac{\bar{C}}{\bar{K}} = X$. Equation (35) allows to identify X as a function of g :

$$X = \frac{\xi \phi(\mu, \mu) \bar{P}(g)}{1-\alpha} \left[\frac{1 - e^{A_{\max}(r-\beta-\rho-g)}}{g-r+\beta+\rho} - \frac{\alpha(1 - e^{A_{\max}(r-\rho-g)})}{g-r+\rho} \right] \quad (36)$$

We only need another equation in terms of X and g to identify both, and this equation is simply the resource constraint of the economy. Then, combining (36) and (16), one can single out a scalar equation involving only the growth rate g :

$$F(g, \Phi) = BL^{1-\varepsilon} - g - \frac{\xi \phi(\mu, \mu) \bar{P}(g)}{1-\alpha} \left[\frac{1 - e^{A_{\max}(r-\beta-\rho-g)}}{g-r+\beta+\rho} - \frac{\alpha(1 - e^{A_{\max}(r-\rho-g)})}{g-r+\rho} \right] = 0 \quad (37)$$

which allows to state the following fundamental proposition:

Proposition 11 *If $g > 0$ exists, then g solves the equation:*

$$F(g; \Phi) = 0 \quad (38)$$

where Φ is the set of parameters, $\Phi = (\alpha; \beta; \xi; \rho; \varepsilon; B)$

The next proposition exhibits a sufficient condition for the g -equation to admit at least a strictly positive root.

Proposition 12 *If L or B large enough, then $g > 0$ solution to $F(\cdot) = 0$ exists*

Proof. See appendix. ■

The sufficient condition is rather standard in economic theory: as in the original Romer (1986) model, a large enough labor force L and/or a large enough productivity parameter B are sufficient to obtain positively sloped balanced growth paths. So in a sense, and since our model relies partly on Romer's specifications, it is good news. Unfortunately, it has been impossible to bring out any analytical appraisal of the uniqueness issue. As one can see, our nonlinear equation $F(g, \cdot) = 0$ is terrific: have in mind that even the single term $\bar{P}(g)$ is a complicated function of g ! Compared to the modified model of Section 4 with age-dependent earnings, we have not only high degree g -polynomials to deal with but also several exponential terms depending on g and arising from the finite maximal life property of the assumed survival law. So studying uniqueness analytically is simply unbearable. Instead, we resort to numerical simulations with the necessary corroborating sensitivity tests. In all the considered (numerous) parameterizations, the g -equation has a unique strictly positive solution. Then, we studied how this solution varies when the demographic parameters change both theoretically and numerically.

5.5 The relationship between economic growth and longevity explored

We start with an analytical result showing that in contrast to the benchmark growth model with perpetual youth, the relationship between growth and longevity cannot be strictly concave. We prove that it should be convex for low values of life expectancy and surely concave if life expectancy is large enough.¹³

Proposition 13 *For α small enough, g is an increasing convex function of α . In contrast, g is necessarily a concave function of α when this parameter is large enough.*

Proof. The second part of the proposition is intuitive. It simply derives from the fact that since the long-run growth rate g is bounded, the increment of g following an increase in α should start decreasing after some value of α large enough. Indeed by (16), we can deduce that $g \leq B$, with B a productivity parameter independent of α . This implies that the growth rate of g should turn to negative (or zero) when α keeps growing, which disqualifies any strict convexity for large α values.

The first part of the proposition is definitely much trickier and its detailed proof is reported in the appendix. ■

Figures 16 to 19 complete and illustrate our proposition. There are principally two findings.

Insert Figures 16 to 19

¹³It might be surprising that with an apparently much more complicated analytical problem than in Section 4, we can provide a finer analytical characterization of the relationship between growth and life expectancy. Actually, the latter is captured via the parameter α in the Proposition below, which turns out to be much easier to deal with than parameter p in Section 4. Of course, the counterpart of parameter p in the modified survival law is rather the exponent β , and we can neither bring out any analytical result with respect to this parameter in this Section.

1. In all our experiments, the growth rate g is an increasing function of longevity. When either α or β increases, the economic growth rate also increase. Recall the mechanisms at work. In our model, an individual saves definitely much less when old compared to her youth. Therefore, if economic development relies on accumulation of physical capital as in our model, and if such an accumulation is only possible thanks to domestic savings, then a larger longevity has also a negative effect on growth by increasing the proportion of people with relatively small saving rates. Our simulations show that at least for the set of reasonable parameterizations considered such a negative impact of increasing longevity is not enough to offset its positive contributions to growth.
2. Nonetheless, one can notice that the shape of the growth rate g as a function is mostly convex-concave, which is consistent with our empirical study, and specially with Figures 5 and 6. One may notice that such a property does not appear in Figure 18, the shape is globally concave. However, it should be noted that in this figure, the maximal age ranges from 109 to 217, and life expectancy ranges from 64 to 145, and convexity only appears when these longevity measures are low enough. Such a claim is reinforced by our Figure 19 which has a shape very similar to the estimated relationship in Figures 5 and 6. What it is the rationale behind? Well, the story is quite simple: when life expectancy is low (and economic growth is low), a further increase in life expectancy is likely to be effective in raising growth through an increment in aggregate savings, which explains why the curve is convex for low values of α . However, if life expectancy is already high, the increment in growth resulting from a further increase in life expectancy is likely to be softened because of the important proportion of elderly whose saving rates are low.

6 Conclusion

In this paper, we have studied the relationship between economic growth and longevity in a model with different demographic structures and with endogenous growth. We have started with a nonparametric econometric appraisal of this relationship on historical data showing a globally increasing but convex-concave shape. We show that life-cycle behavior combined with age-dependent survival laws can reproduce such an empirical finding while age-dependent earning in the spirit of Faruquee et al. (1997) rather produce the more popular hump-shaped relationship between longevity and economic growth. In our theory, while the economic growth rate is an increasing function of the life expectancy parameter α , its first-order derivative is non-monotonic, reflecting the growth enhancing effect of longevity at low levels of development and longevity.

Two interesting extensions are currently in our agenda. One natural idea is to try to endogenize life expectancy via public and private health expenditures. This could be done by considering that either parameter α or β (or both) does depend on such expenditures. This would provide a richer (and more realistic picture) of the relationship between life expectancy and economic growth. Admittedly, a substantial part of the rise in longevity registered in the twentieth century is due to rising (and more efficient) health expenditures, which was in turn made possible by better economic conditions. Incorporating health expenditures in our set-up will then result in a better and more precise appraisal of the relationship between longevity

and economic growth. Unfortunately, such an extension is not trivial as it involves (notably via the endogenization of β) further mathematical difficulties (like endogenous discounting, time inconsistency,...), which are not that easy to tackle within a vintage structure like ours. Another valuable extension would consist in replacing our discussion in a broader demographic transition set-up more tightly connected to the historical data we have used in the second section of the paper. This requires in particular endogenizing fertility (or equivalently, the population growth parameter, n , of Section 3). This is a more challenging task which deserves a particular attention.

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7 Appendix

Proof of Proposition 6

By integrating Equation (6) and the transversality condition, the budgetary constraint can be rewritten as follows

$$\begin{aligned}
\int_z^\infty C(\mu, v) e^{-\int_z^v [r(\epsilon) + S(\epsilon - z)] d\epsilon} dv &= R(\mu, z) + D(\mu, z) \\
\int_\mu^\infty C(\mu, \mu) e^{\int_\mu^v (r(s) - \rho) ds} e^{-\int_\mu^v (r(\epsilon) + S(\epsilon - \mu)) d\epsilon} dv &= R(\mu, \mu) + D(\mu, \mu) \\
C(\mu, \mu) \int_\mu^\infty e^{-\int_\mu^v (\rho + S(x - \mu)) dx} dv &= R(\mu, \mu) + D(\mu, \mu) \\
\Rightarrow C(\mu, \mu) &= \phi(\mu, \mu) [R(\mu, \mu) + D(\mu, \mu)]
\end{aligned}$$

with

$$\phi(\mu, \mu) = \frac{1}{\int_\mu^\infty e^{-\int_\mu^v [\rho + S(\alpha, \beta, (x - \mu))] dx} dv} \quad (39)$$

using the budgetary constraint we obtain that the quantity

$$\int_z^\infty C(\mu, \mu) e^{\int_\mu^z (r(s) - \rho) ds} e^{\int_z^v (r(s) - \rho) ds} e^{-\int_z^v (r(\epsilon) + S(\epsilon - z)) d\epsilon} dv$$

turns out to be:

$$\begin{aligned}
C(\mu, \mu) e^{\int_\mu^z (r(s) - \rho) ds} \int_z^\infty e^{-\int_z^v (\rho + S(s - z)) ds} dv &= R(\mu, z) + D(\mu, z) \\
\Rightarrow C(\mu, z) &= \phi(\mu, z) [R(\mu, z) + D(\mu, z)] \quad \blacksquare
\end{aligned}$$

Proof of Proposition 7

We obtain:

$$\begin{aligned}
\phi(\mu, z) &= \frac{1}{\int_z^\infty e^{-\int_z^v [\rho + S(\alpha, \beta, (x - \mu))] dx} dv} \\
\phi(\mu, z)^{-1} &= \int_z^\infty e^{-\int_z^v [\rho + S(\alpha, \beta, (x - \mu))] dx} dv
\end{aligned}$$

Now, since $\int_z^v [\rho + S(\alpha, \beta, (x - \mu))] dx = \rho(v - z) - \ln \left(\frac{e^{-\beta(v - \mu)} - \alpha}{e^{-\beta(z - \mu)} - \alpha} \right)$, then

$$\begin{aligned}
\phi(\mu, z)^{-1} &= \int_z^\infty e^{-\rho(v - z) + \ln \frac{e^{-\beta(v - \mu)} - \alpha}{e^{-\beta(z - \mu)} - \alpha}} dv \\
&= \int_z^\infty e^{-\rho(v - z)} \frac{e^{-\beta(v - \mu)} - \alpha}{e^{-\beta(z - \mu)} - \alpha} dv \\
&= \frac{1}{e^{-\beta(z - \mu)} - \alpha} \left[\frac{e^{-\beta(z - \mu)}}{\rho + \beta} - \frac{\alpha}{\rho} \right] \\
&= \frac{\rho e^{-\beta(z - \mu)} - \alpha(\rho + \beta)}{\rho(e^{-\beta(z - \mu)} - \alpha)(\rho + \beta)}
\end{aligned}$$

Finally the marginal propensity to consume at z for an individual born at μ is:

$$\phi(\mu, z) = \frac{\rho(\rho + \beta)(e^{-\beta(z-\mu)} - \alpha)}{\rho e^{-\beta(z-\mu)} - \alpha(\rho + \beta)} \quad (40)$$

The marginal propensity to consume for a newly born individual is:

$$\phi(\mu, \mu) = \frac{1}{\int_{\mu}^{\infty} e^{-\int_{\mu}^v [\rho + S(\alpha, \beta, (x-\mu))] dx} dv}$$

and,

$$\begin{aligned} \phi(\mu, \mu)^{-1} &= \int_{\mu}^{\infty} e^{-\rho(v-z) + \ln \frac{e^{-\beta(v-\mu)} - \alpha}{e^{-\beta(\mu-\mu)} - \alpha}} dv \\ &= \int_{\mu}^{\infty} e^{-\rho(v-z)} \frac{e^{-\beta(v-\mu)} - \alpha}{1 - \alpha} dv \\ &= \frac{[e^{-(v-\mu)(\beta+\rho)}]_{\mu}^{\infty}}{(1-\alpha)(\rho+\beta)} + \frac{\alpha [e^{-\rho(v-\mu)}]_{\mu}^{\infty}}{\rho(1-\alpha)} \\ &= \frac{1}{(1-\alpha)(\rho+\beta)} - \frac{\alpha}{\rho(1-\alpha)} \\ &= \frac{\rho - \alpha(\rho + \beta)}{\rho(1-\alpha)(\rho + \beta)} \end{aligned}$$

Then,

$$\phi(\mu, \mu) = \frac{\rho(1-\alpha)(\rho + \beta)}{\rho - \alpha(\rho + \beta)} \quad (41)$$

Now, let us study the evolution of $\phi(\mu, \mu)$ with respect to β , α and ρ .

$$\begin{aligned} \frac{\partial \phi(\mu, \mu)}{\partial \beta} &= \frac{\rho(1-\alpha) [\rho - \alpha(\rho + \beta)] + \alpha \rho(1-\alpha)(\rho + \beta)}{[\rho - \alpha(\rho + \beta)]^2} \\ &= \frac{\rho^2(1-\alpha) - \alpha \rho(1-\alpha)(\rho + \beta) + \alpha \rho(1-\alpha)(\rho + \beta)}{[\rho - \alpha(\rho + \beta)]^2} \end{aligned}$$

Then

$$\frac{\partial \phi(\mu, \mu)}{\partial \beta} = \frac{\rho^2(1-\alpha)}{[\rho - \alpha(\rho + \beta)]^2} \quad (42)$$

Since $\alpha > 1$, then $\frac{\partial \phi(\mu, \mu)}{\partial \beta} < 0$. With the expression of the maximum age we have supposed that $\beta < 0$, then an increase in β yields an increasing consumption.

$$\begin{aligned} \frac{\partial \phi(\mu, \mu)}{\partial \alpha} &= \frac{-\rho(\rho + \beta) [\rho - \alpha(\rho + \beta)] + \rho(\rho + \beta)^2(1-\alpha)}{[\rho - \alpha(\rho + \beta)]^2} \\ &= \frac{-\rho^2(\rho + \beta) + \alpha \rho(\rho + \beta)^2 - \alpha \rho(\rho + \beta)^2 + \rho(\rho + \beta)^2}{[\rho - \alpha(\rho + \beta)]^2} \\ &= \frac{-\rho^2(\rho + \beta) + \rho(\rho + \beta)^2}{[\rho - \alpha(\rho + \beta)]^2} \\ &= \frac{\rho(\rho + \beta) (-\rho + \rho + \beta)}{[\rho - \alpha(\rho + \beta)]^2} \end{aligned}$$

Then,

$$\frac{\partial \phi(\mu, \mu)}{\partial \alpha} = \frac{\beta \rho(\rho + \beta)}{[\rho - \alpha(\rho + \beta)]^2} \quad (43)$$

Since $\beta < 0$, then $\frac{\partial \phi(\mu, \mu)}{\partial \alpha} < 0$ if and only if $\rho + \beta > 0$. That is to say $\rho > |\beta|$.

The evolution of the marginal propensity to consume with respect to the intertemporal discount rate governed by

$$\begin{aligned} \frac{\partial \phi(\mu, \mu)}{\partial \rho} &= \frac{[(1 - \alpha)(\rho + \beta) + \rho(1 - \alpha)][\rho - \alpha\rho(\rho + \beta)] - \rho(1 - \alpha)^2(\rho + \beta)}{[\rho - \alpha(\rho + \beta)]^2} \\ &= \frac{[(1 - \alpha)(\rho + \beta)][\rho - \alpha\rho(\rho + \beta)] + \rho^2(1 - \alpha) - \rho(1 - \alpha)(\rho + \beta)}{[\rho - \alpha(\rho + \beta)]^2} \end{aligned}$$

Then,

$$\frac{\partial \phi(\mu, \mu)}{\partial \rho} = \frac{(1 - \alpha)[\rho^2 - \alpha(\rho + \beta)^2]}{[\rho - \alpha(\rho + \beta)]^2} \quad (44)$$

An increasing ρ means that agent prefer the present consumption; then they have a decreasing saving rate. That is to say $\frac{\partial \phi(\mu, \mu)}{\partial \rho} > 0$. For this purpose, we must have $\rho^2 - \alpha(\rho + \beta)^2 < 0$, since $\alpha > 1$. Finally, $\rho^2 - \alpha(\rho + \beta)^2 < 0 \Rightarrow \rho^2 < \alpha(\rho + \beta)^2$ and $\alpha > \frac{\rho^2}{(\rho + \beta)^2}$. ■

Proof of Proposition 9

We have,

$$\begin{aligned} D(z) &= \int_{-\infty}^z D(\mu, z)T(\mu, z)d\mu \quad \text{with} \quad D(\mu, z) = \int_z^\infty w(v)e^{-\int_z^v [r(s) + S(s - \mu)]ds}dv \\ \dot{D}(z) &= \xi D(z, z) + \int_{-\infty}^z \dot{D}(\mu, z)T(\mu, z)d\mu + \int_{-\infty}^z D(\mu, z)\dot{T}(\mu, z)d\mu \\ \dot{D}(\mu, z) &= -w(z) + \int_z^\infty \dot{w}(v)e^{-\int_z^v [r(s) + S(s - \mu)]ds}dv + \\ &\quad \int_z^\infty w(v)[r(z) + S(z - \mu)]e^{-\int_z^v [r(s) + S(s - \mu)]ds}dv \end{aligned}$$

Using $\dot{T}(\mu, z) = -S(z - \mu)T(\mu, z)$ the above relation is computed as

$$\begin{aligned} \dot{D}(z) &= \xi D(z, z) + \int_{-\infty}^z \left[-w(z) + \int_z^\infty \dot{w}(v)e^{-\int_z^v [r(s) + S(s - \mu)]ds}dv \right. \\ &\quad \left. + \int_z^\infty w(v)[r(z) + S(z - \mu)]e^{-\int_z^v [r(s) + S(s - \mu)]ds}dv \right] T(\mu, z)d\mu \\ &\quad + \int_{-\infty}^z D(\mu, z)\dot{T}(\mu, z)d\mu \end{aligned}$$

which implies that,

$$\begin{aligned} \dot{D}(z) &= \xi D(z, z) - w(z)L(z) \\ &\quad + \int_{-\infty}^z \int_z^\infty w(v)[r(z) + S(z - \mu)]e^{-\int_z^v [r(s) + S(s - \mu)]ds}dv T(\mu, z)d\mu \\ &\quad - \int_{-\infty}^z D(\mu, z)T(\mu, z)S(z - \mu)d\mu \quad \blacksquare \end{aligned}$$

Proof of Proposition 10

We have,

$$\begin{aligned}
R(z) &= \int_{-\infty}^z R(\mu, z)T(\mu, z)d\mu \\
\dot{R}(z) &= R(z, z)T(z, z) + \int_{-\infty}^z \dot{R}(\mu, z)T(\mu, z)d\mu + \int_{-\infty}^z R(\mu, z)\dot{T}(\mu, z)d\mu \\
&= R(z, z)T(z, z) + \int_{-\infty}^z \dot{R}(\mu, z)T(\mu, z)d\mu - \int_{-\infty}^z R(\mu, z)S(\mu, z)T(\mu, z)d\mu
\end{aligned}$$

Using the budget constraint given by the equation

$$\begin{aligned}
\dot{R}(z) &= R(z, z)T(z, z) + \int_{-\infty}^z [(r(z) + S(\mu, z))R(\mu, z) + w(z) - c(\mu, z)]T(\mu, z)d\mu \\
&\quad - \int_{-\infty}^z R(\mu, z)S(\mu, z)T(\mu, z)d\mu
\end{aligned}$$

we obtain

$$\dot{R}(z) = R(z, z)T(z, z) + r(z)R(z) + w(z)L(z)d\mu - C(z) \quad \blacksquare$$

Proof of Proposition 12

We have $\lim_{g \rightarrow +\infty} F(g) = -\infty$ and,

$$\begin{aligned}
F(0) &= BL^{1-\alpha} - \frac{\xi\phi(\mu, \mu)(1-\varepsilon)BL^{-\varepsilon} \frac{1}{1-\alpha} \left[\frac{e^{A_{\max}(-r-\beta)}-1}{-r-\beta} - \frac{\alpha(e^{A_{\max}(-r)}-1)}{-r} \right]}{1-\alpha} \\
&\quad \times \left[\frac{1 - e^{A_{\max}(r-\beta-\rho)}}{-r+\beta+\rho} - \frac{\alpha(1 - e^{A_{\max}(r-\rho)})}{-r+\rho} \right] \\
\lim_{B, L \rightarrow \infty} F(0) &= \lim_{B, L \rightarrow \infty} BL^{1-\varepsilon} - \xi\phi(\mu, \mu)1 - \varepsilon BL^{-\varepsilon} \left[\frac{e^{-A_{\max}\varepsilon BL^{1-\varepsilon}}e^{-\beta}}{-(1-\alpha)^2\varepsilon BL^{1-\varepsilon}} + \frac{\alpha e^{-A_{\max}\varepsilon BL^{1-\varepsilon}}}{(1-\alpha)^2\varepsilon BL^{1-\varepsilon}} \right] \\
&\quad \times \left[\frac{-e^{A_{\max}\varepsilon BL^{1-\varepsilon}}e^{-\beta-\rho}}{-\varepsilon BL^{1-\varepsilon}} - \frac{\alpha e^{A_{\max}\varepsilon BL^{1-\varepsilon}}e^{-\rho}}{\varepsilon BL^{1-\varepsilon}} \right] \\
&= \lim_{B, L \rightarrow \infty} BL^{1-\varepsilon} - \xi\phi(\mu, \mu)1 - \varepsilon BL^{-\varepsilon} \left[\frac{-e^{-A_{\max}\varepsilon BL^{1-\varepsilon}}}{(1-\alpha)^2\varepsilon BL^{1-\varepsilon}} (e^{-\beta} - \alpha) \right] \\
&\quad \times \left[\frac{e^{A_{\max}\varepsilon BL^{1-\varepsilon}}}{\varepsilon BL^{1-\varepsilon}} (e^{-\beta-\rho} - \alpha e^{-\rho}) \right] \\
\lim_{B, L \rightarrow \infty} F(0) &= \lim_{B, L \rightarrow \infty} BL^{1-\varepsilon} + \xi\phi(\mu, \mu)1 - \varepsilon BL^{-\varepsilon} \left[\frac{e^{A_{\max}\varepsilon BL^{1-\varepsilon}(1-1)}}{(1-\alpha)^2(\varepsilon BL^{1-\varepsilon})^2} \right] (e^{-\beta} - \alpha) (e^{-\beta-\rho} - \alpha e^{-\rho}) \\
&= \lim_{B, L \rightarrow \infty} BL^{1-\varepsilon} + \left[\frac{\xi\phi(\mu, \mu)(1-\varepsilon)BL^{-\varepsilon} (e^{-\beta} - \alpha) (e^{-\beta-\rho} - \alpha e^{-\rho})}{(1-\alpha)^2(\varepsilon BL^{1-\varepsilon})^2} \right] \\
&= \lim_{B, L \rightarrow \infty} BL^{1-\varepsilon} + \left[\frac{\xi\phi(\mu, \mu)(1-\varepsilon) (e^{-\beta} - \alpha) (e^{-\beta-\rho} - \alpha e^{-\rho})}{(1-\alpha)^2\varepsilon^2 BL^{2-\varepsilon}} \right]
\end{aligned}$$

Then,

$$\lim_{B \rightarrow \infty} F(0) = +\infty$$

and,

$$\lim_{L \rightarrow \infty} F(0) = +\infty \text{ since } \varepsilon < 1 \quad \blacksquare$$

Proof of Proposition 13

We are considering here the first part of Proposition (13). It follows that:

$$\begin{aligned} F &= BL^{1-\varepsilon} - g + \frac{\xi\phi(\mu, \mu)\bar{G}}{(1-\alpha)^2 D(g)} \left[\left(e^{A_{\max}(g-r-\beta)} - 1 \right) (g-r) - \alpha \left(e^{A_{\max}(g-r)} - 1 \right) (g-r-\beta) \right] \\ &\quad \times \left[(g-r+\rho) \left(1 - e^{A_{\max}(r-\beta-\rho-g)} \right) - \alpha \left(1 - e^{A_{\max}(r-\rho-g)} \right) (g-r+\beta+\rho) \right] \end{aligned}$$

Since $\lim_{\alpha \rightarrow 1} A_{\max} = 0$ then we can approximate $e^{A_{\max}(\cdot)}$ by $1 + A_{\max}(\cdot)$, then the function F can be rewritten as

$$F = BL^{1-\varepsilon} - g + \frac{\xi\phi(\mu, \mu)\bar{G}}{D(g)} A_{\max}^2 (g-r-\beta) (g-r) (r-\rho-g) (r-\beta-\rho-g)$$

With $D(g) = g^4 + g^3(-4r+2\rho) + g^2[r(5r+\beta-4\rho-2) - \beta(\beta+2\rho-1) + 2\rho] + g(-2r+2\rho+\beta)(-2r+\beta+r^2+\beta r) + (r^2+\beta r)(-2r+2\rho+\beta)$

$$R = (g-r-\beta)(g-r)(r-\rho-g)(r-\beta-\rho-g)$$

$S = \left(\frac{D'(g)}{D(g)}(g-r) - 1 \right) (g-r-\beta)(g-r+\rho)(r-\rho-\beta-g) - (g-r)(r-\rho-\beta-g)(-2g+2r-2\rho-\beta) + (g-r+\rho)(g-r-\beta)(g-r)$ Moreover, we can rewrite F as

$$F = (BL^{1-\varepsilon} - g) D(g) + \xi\phi(\mu, \mu)\bar{G} A_{\max}^2 (g-r-\beta) (g-r) (r-\rho-g) (r-\beta-\rho-g) \quad (45)$$

Now, observe that if $\lim_{\alpha \rightarrow 1} A_{\max} = 0$, then either L or g tend to zero. Consequently we can just consider the polynomes of degree one in g as first-order approximations. From the Equation (45) we have

$$\begin{aligned} A_{\max}^2 &= \frac{(BL^{1-\varepsilon} - g) D(g)}{\xi\phi(\mu, \mu)\bar{G}(g-r-\beta)(g-r)(r-\rho-g)(r-\beta-\rho-g)} \\ &= \frac{(BL^{1-\varepsilon} - g)(r^2 + \beta r)(-2r + 2\rho + \beta)}{\xi\phi(\mu, \mu)\bar{G}(g-r-\beta)(g-r)(r-\rho-g)(r-\beta-\rho-g)} \\ &\quad + \frac{BL^{1-\varepsilon} g(-2r + 2\rho + \beta)(-2r + \beta + r^2 + \beta r)}{\xi\phi(\mu, \mu)\bar{G}(g-r-\beta)(g-r)(r-\rho-g)(r-\beta-\rho-g)} \\ A_{\max}^2 &\approx \xi\phi(\mu, \mu)\bar{G} BL^{1-\varepsilon} (r^2 + \beta r)^2 (-2r + 2\rho + \beta) (r^2 - r(\beta + 2\rho) + \rho^2 + \beta\rho) \\ &\quad + g\xi\phi(\mu, \mu)\bar{G} BL^{1-\varepsilon} [(-2r - \beta) [(r^2 - r(\beta + 2\rho) + \rho^2 + \beta\rho) + r^2 + \beta r] \times \\ &\quad [(r^2 + \beta r)(-2r + 2\rho + \beta)] + 2[2r(r - \beta - \rho) + \beta + \rho] \times \\ &\quad [(r^2 + \beta r)(-r^2 + r(\beta + 2\rho) - \rho^2 - \beta\rho)]] \\ A_{\max}^2 &\approx \xi\phi(\mu, \mu)\bar{G} BL^{1-\varepsilon} (r^2 + \beta r)^2 (-2r + 2\rho + \beta) (r^2 - r(\beta + 2\rho) + \rho^2 + \beta\rho) \\ &\quad + g\xi\phi(\mu, \mu)\bar{G} BL^{1-\varepsilon} M \end{aligned}$$

where $M = (-2r - \beta) [(r^2 - r(\beta + 2\rho) + \rho^2 + \beta\rho) + r^2 + \beta r] \times$

$$[(r^2 + \beta r)(-2r + 2\rho + \beta)] + 2[2r(r - \beta - \rho) + \beta + \rho] [(r^2 + \beta r)(r^2 - r(\beta + 2\rho) + \rho^2 + \beta\rho)]$$

We can determine now the function $g(A_{\max})$:

$$g(A_{\max}) = \frac{A_{\max}^2}{\xi\phi(\mu, \mu)\tilde{G}BL^{1-\varepsilon}M} - \frac{(r^2 + \beta r)^2(-2r + 2\rho + \beta)(-r^2 + r(\beta + 2\rho) - \rho^2 - \beta\rho)}{M}$$

Furthermore,

$$\begin{aligned} g(A_{\max}) &= \frac{A_{\max}^2}{\xi\phi(\mu, \mu)\tilde{G}BL^{1-\varepsilon}M} \\ &= \frac{A_{\max}^2}{\xi\phi(\mu, \mu)(1 - \varepsilon)L^{-\varepsilon}B^2L^{1-\varepsilon}M} \\ &= \frac{A_{\max}^2}{\xi\phi(\mu, \mu)(1 - \varepsilon)B^2 \left[\frac{\xi}{1-\alpha} \left[\frac{1-e^{-\beta A_{\max}}}{\beta} - \alpha A_{\max} \right] \right]^{1-2\varepsilon} M} \\ &= \frac{(A_{\max})^{2\varepsilon+1}}{\xi^{(2-2\varepsilon)}\phi(\mu, \mu)(1 - \varepsilon)B^2M} \end{aligned}$$

Then

$$\begin{aligned} \frac{\partial g(A_{\max})}{\partial A_{\max}} &= \frac{(2\varepsilon + 1)(A_{\max})^{2\varepsilon}}{\xi^{(2-2\varepsilon)}\phi(\mu, \mu)(1 - \varepsilon)B^2M} \\ \frac{\partial^2 g(A_{\max})}{\partial A_{\max}^2} &= \frac{2\varepsilon(2\varepsilon + 1)(A_{\max})^{2\varepsilon-1}}{\xi^{(2-2\varepsilon)}\phi(\mu, \mu)(1 - \varepsilon)B^2M} \end{aligned}$$

The sign of $\frac{\partial^2 g(A_{\max})}{\partial A_{\max}^2}$ depends on that of M . Since we know that since $\lim_{\alpha \rightarrow 1} A_{\max} = 0$, either L , r or g goes to zero, we ultimately get $M = (2\beta + \rho)(\rho^2 + \beta\rho) > 0$. Also we can conclude that $\frac{\partial^2 g(A_{\max})}{\partial A_{\max}^2} \geq 0$. ■

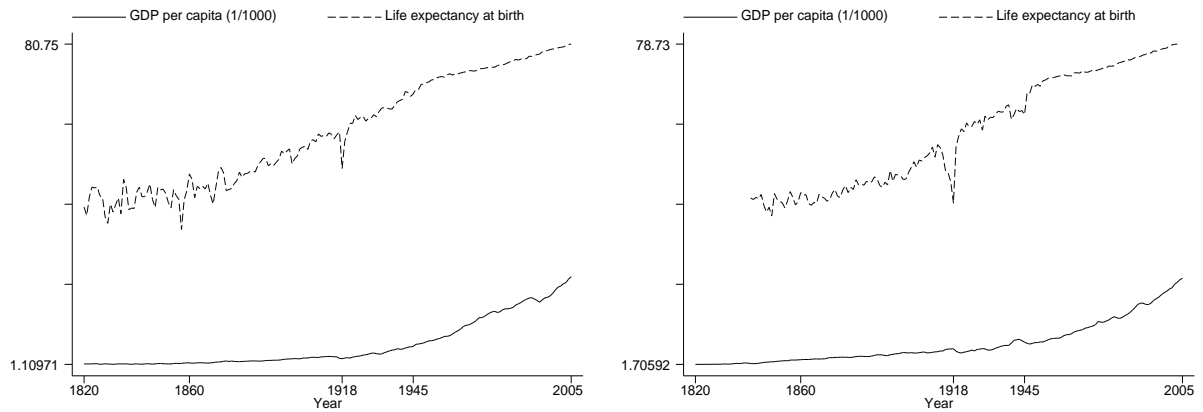


Figure 1: Evolution of GDP per capita and life expectancy at birth: Sweden (left), UK (right)

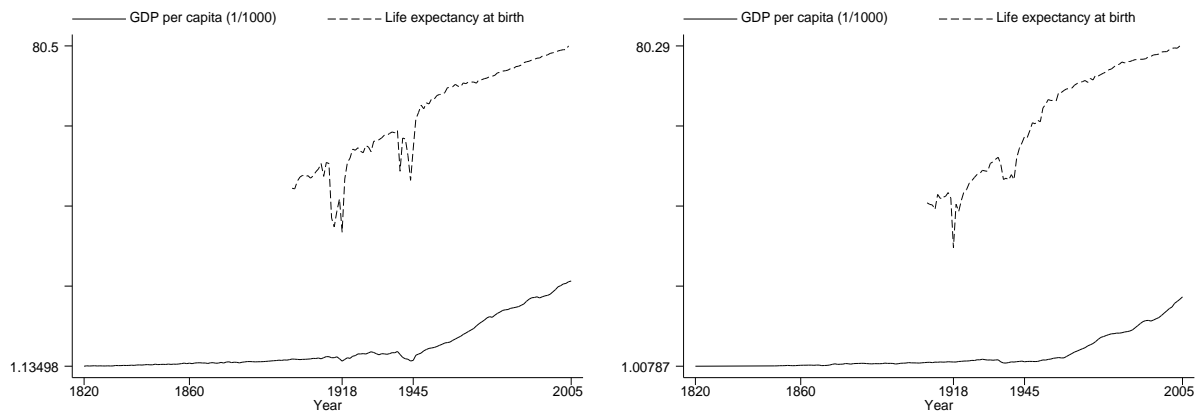


Figure 2: Evolution of GDP per capita and life expectancy at birth: France (left), Spain (right)

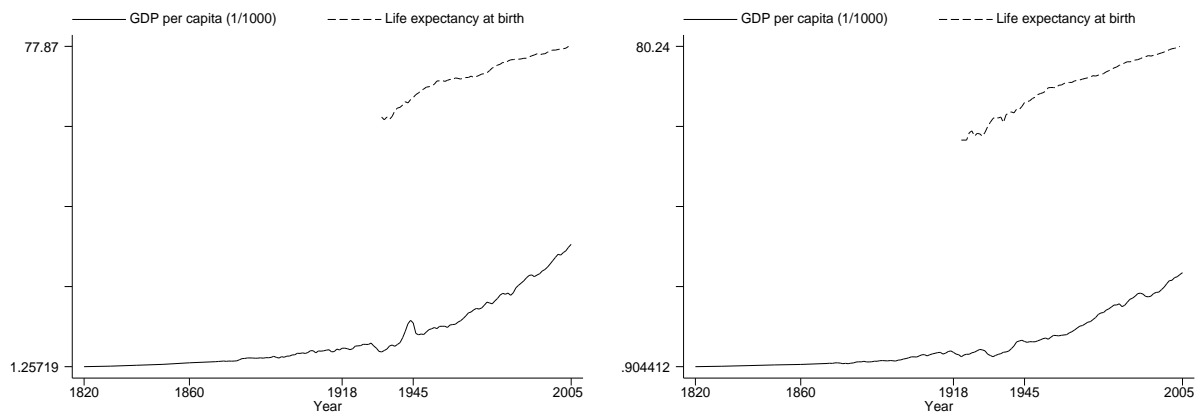


Figure 3: Evolution of GDP per capita and life expectancy at birth: USA (left), Canada (right)

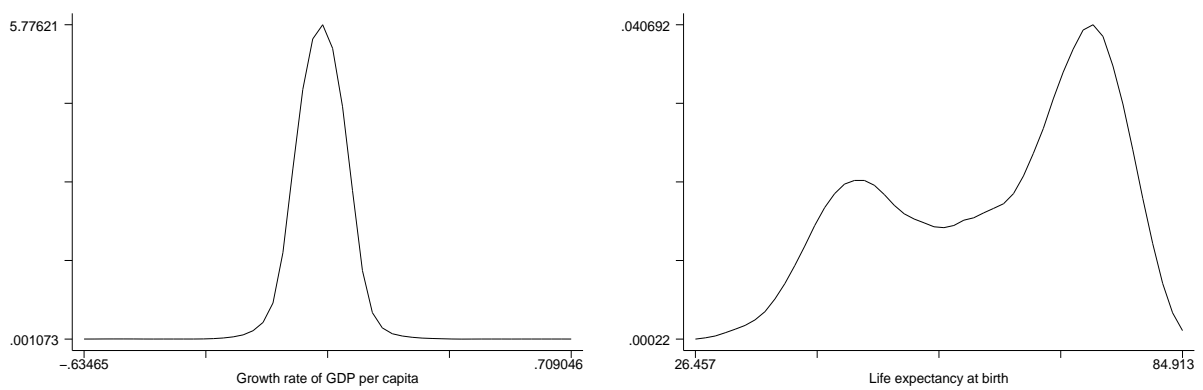


Figure 4: Distribution (nonparametric density estimates) of the growth rate of GDP per capita (left) and life expectancy at birth (right) for the whole sample

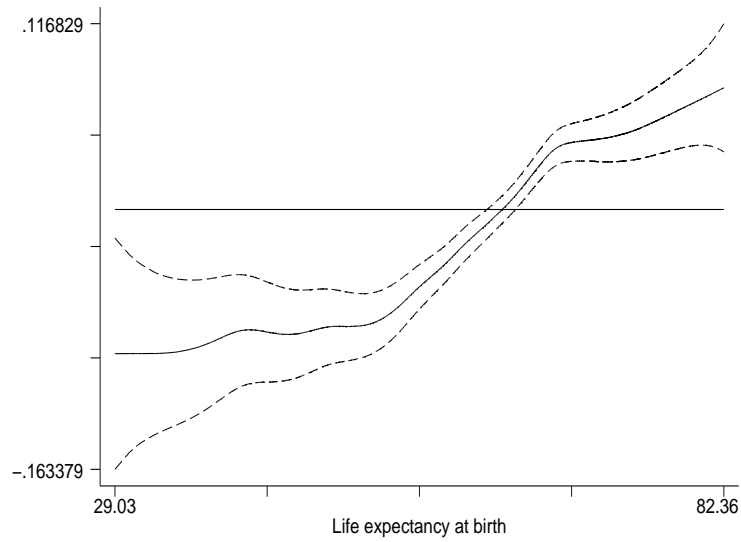


Figure 5: Nonparametric estimation of life expectancy effect on GDP growth rate per capita with yearly data. The solid line represents the nonparametric fit \hat{f} . Dashed lines are 95% bootstrap pointwise confidence intervals. The straight solid line is the zero line.

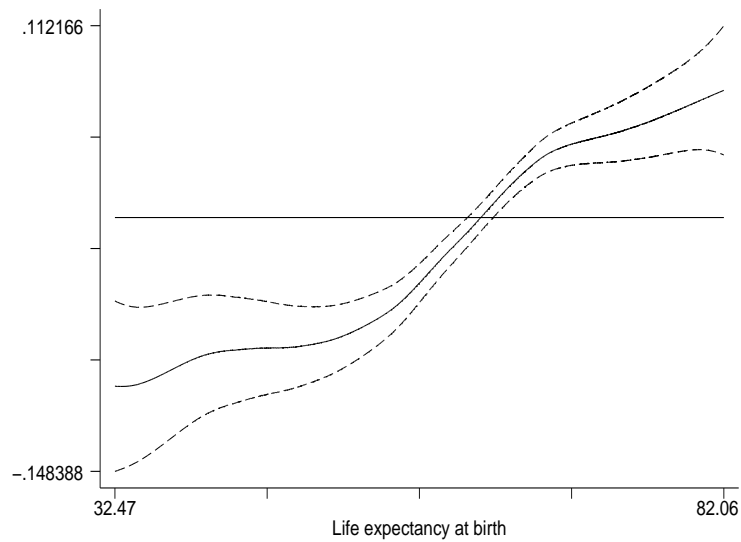


Figure 6: Nonparametric estimation of life expectancy effects on GDP growth rate per capita with 20-years average periods. The solid line represents the nonparametric fit \hat{f} . Dashed lines are 95% bootstrap pointwise confidence intervals. The straight solid line is the zero line.

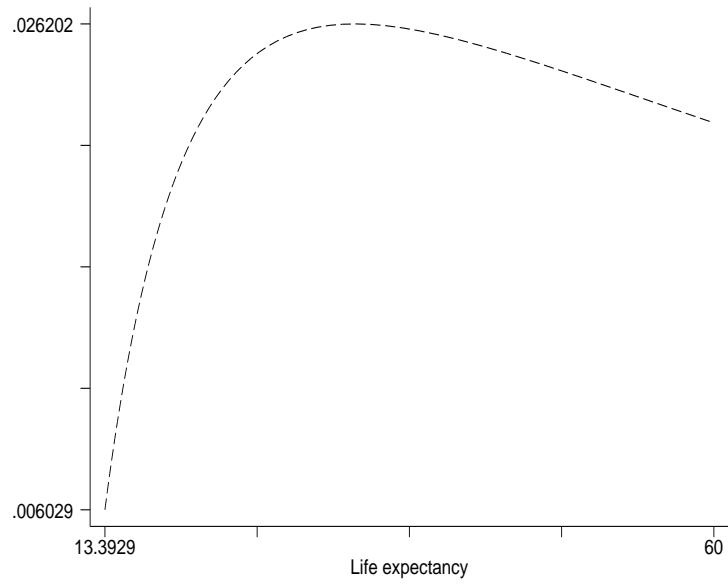


Figure 7: Evolution of the growth rate g with respect to life expectancy.

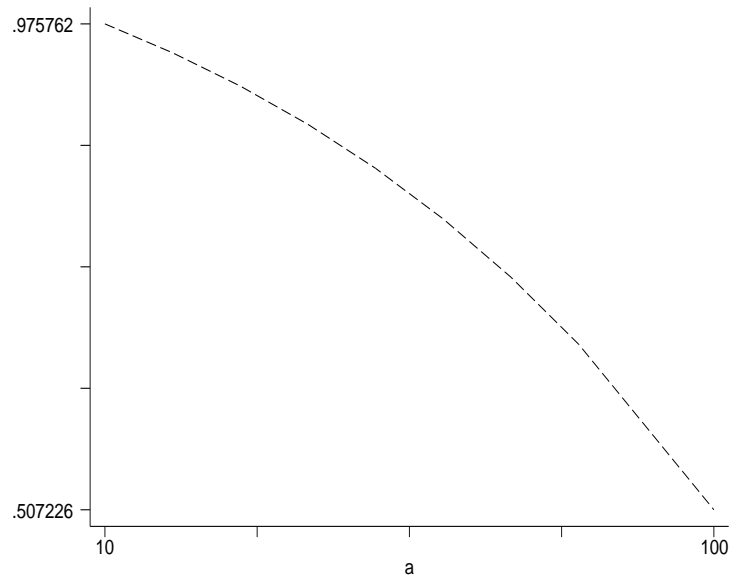


Figure 8: Evolution of survival law $m(a)$ with respect to age (a).
 $\alpha = 5.44$, $\beta = -0.0147$, $E = 73$, $A_{\max} = 115$

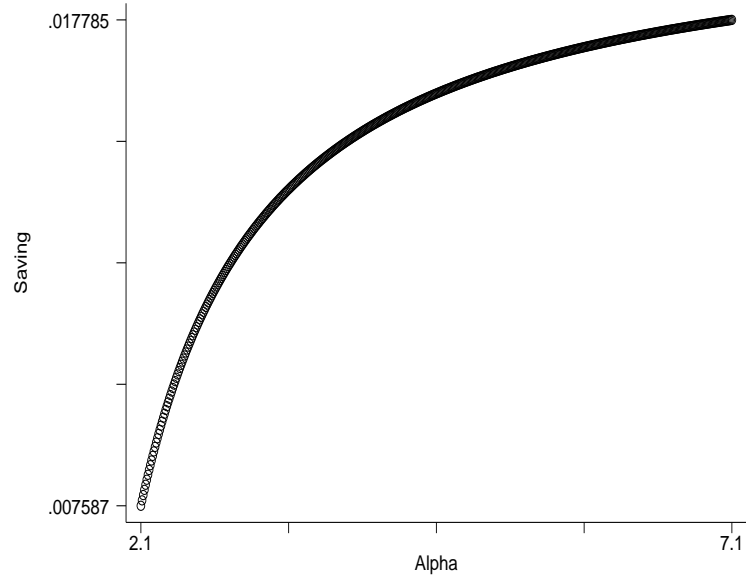


Figure 9: Evolution of $s(\mu, \mu)$ with respect to α .
 $\beta = -0.01502, \rho = 0.98$ and $27.7 < E < 85, 49 < A_{\max} < 130$

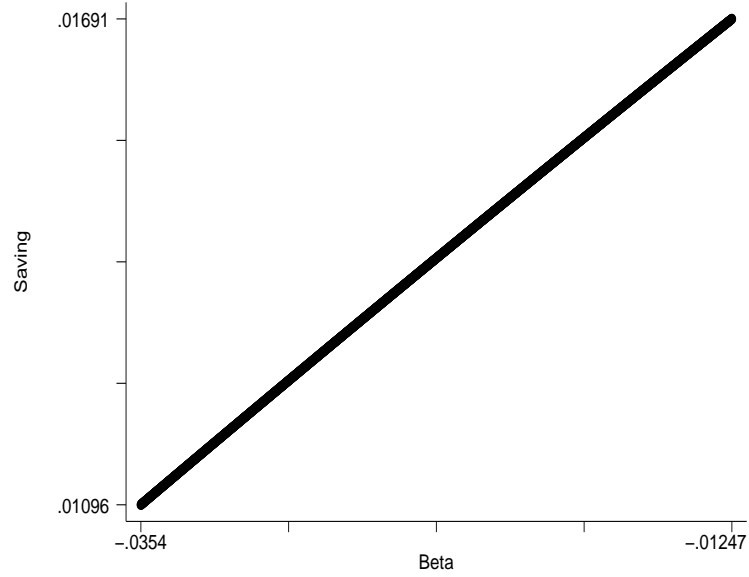


Figure 10: Evolution of $s(\mu, \mu)$ with respect to β .
 $\alpha = 5.1, \rho = 0.98$ and $29.7 < E < 82, 46 < A_{\max} < 130$

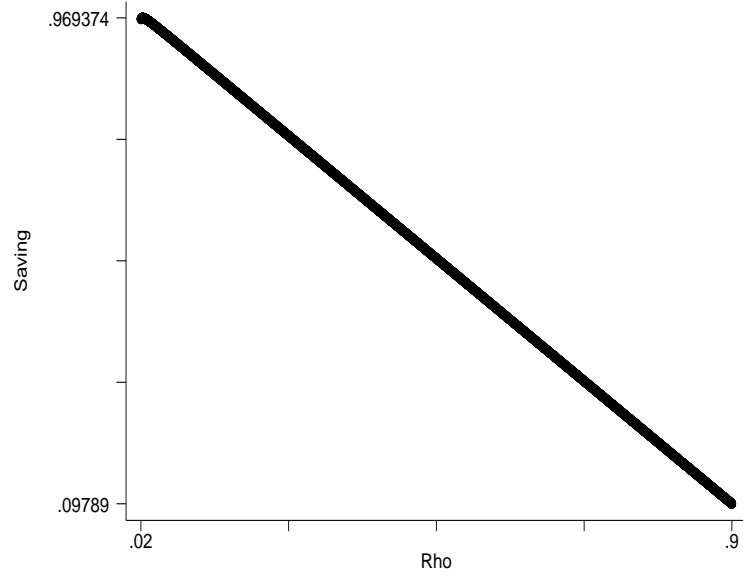


Figure 11: Evolution of $s(\mu, \mu)$ with respect to ρ .
 $\alpha = 8.1, \beta = -0.0147$ and $E = 94, A_{\max} = 142$

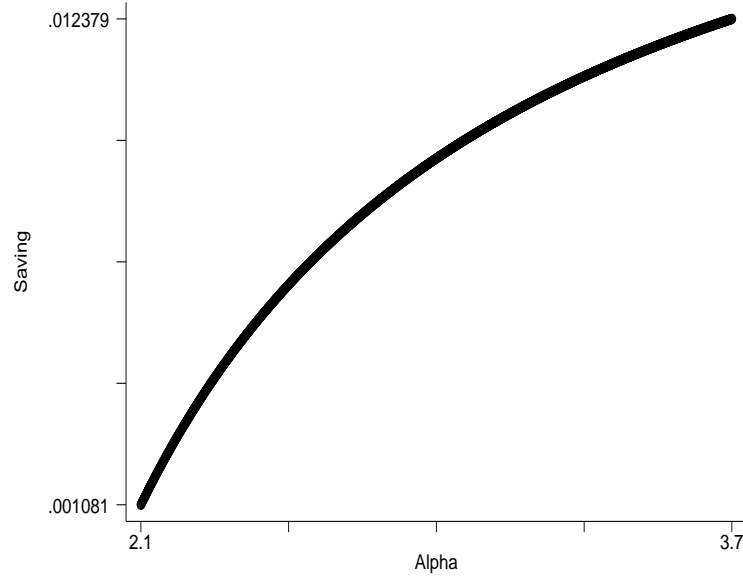


Figure 12: Evolution of $s(\mu, \mu)$ with respect to α .
 $\beta = -0.02, \rho = 0.98$ and $20.8 < E < 39.6, 37.09 < A_{\max} < 66.5$

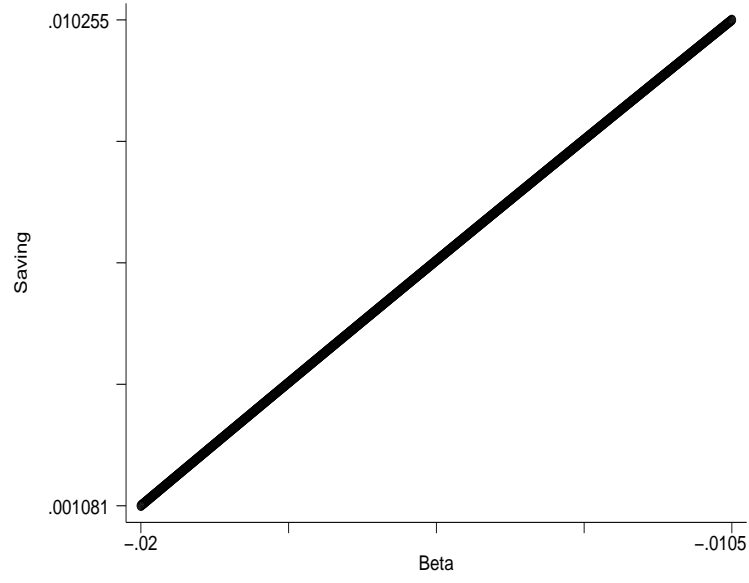


Figure 13: Evolution of $s(\mu, \mu)$ with respect to β .
 $\alpha = 2.1, \rho = 0.98$ and $20.8 < E < 39.65, 37 < A_{\max} < 70.6$

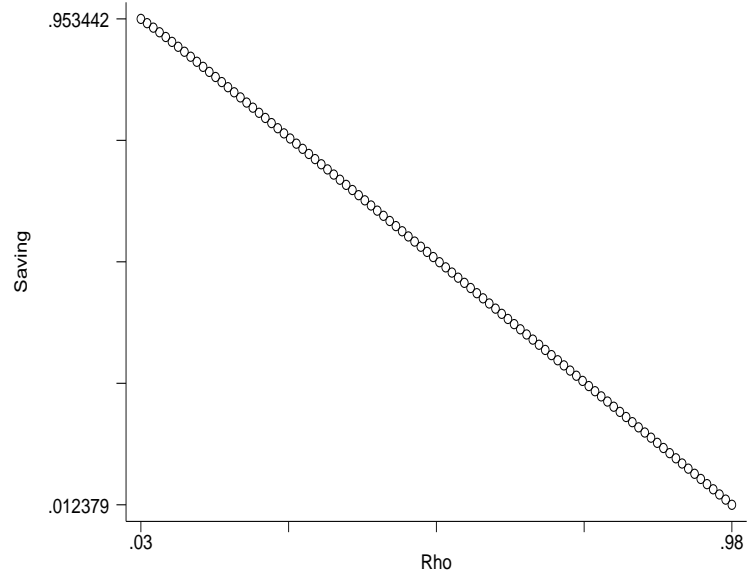


Figure 14: Evolution of $s(\mu, \mu)$ with respect to ρ .
 $\alpha = 3.7, \beta = -0.02$ and $E = 39.6, A_{\max} = 65$

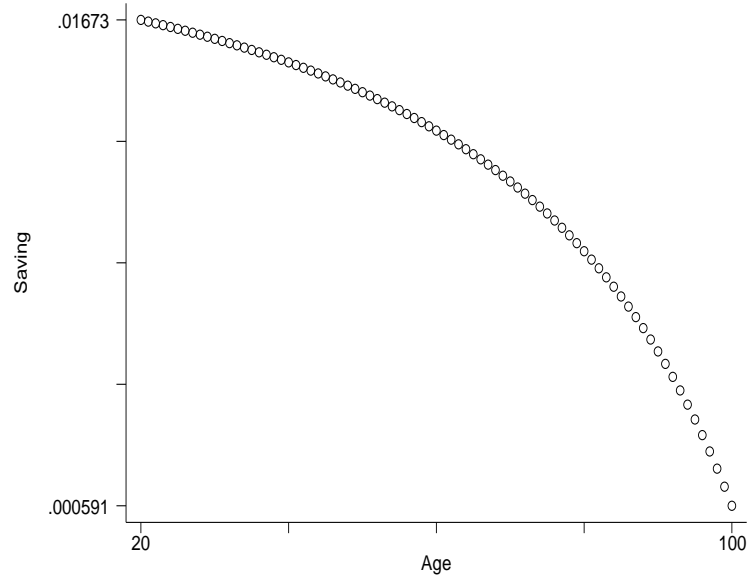


Figure 15: Evolution of saving rate with respect to a .
 $\alpha = 7.1, \beta = -0.01502, \rho = 0.98$ and $E = 91, A_{\max} = 139$

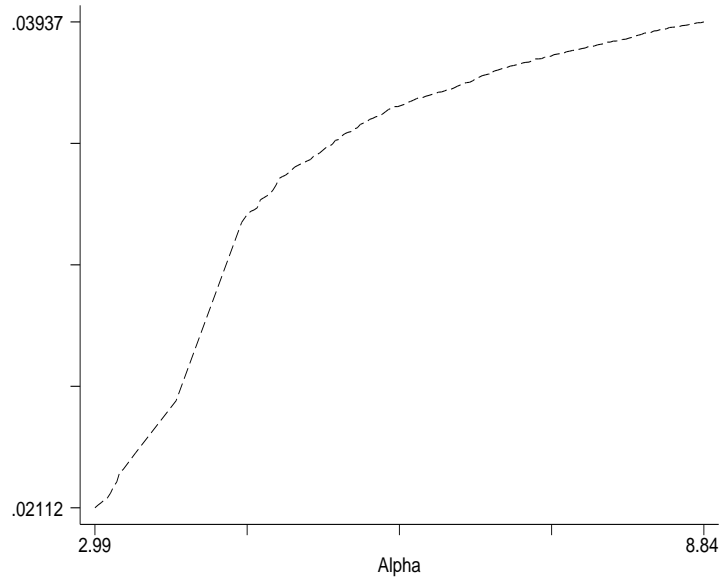


Figure 16: Evolution of g with respect to α .
 $\beta = -0.015, \rho = 0.02$ and $43.03 < E < 97.23, 73 < A_{\max} < 145.3$

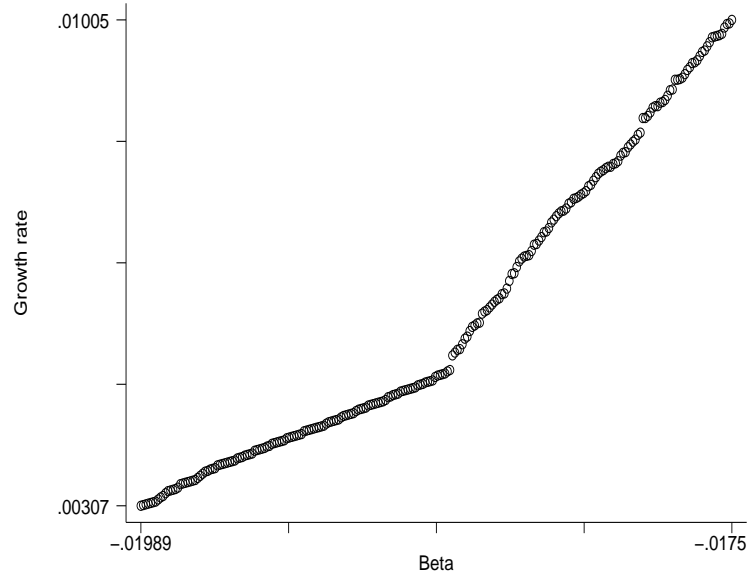


Figure 17: Evolution of g with respect to β .
 $\alpha = 5.1, \rho = 0.02$ and $51 < E < 59, 81.86 < A_{\max} < 93$

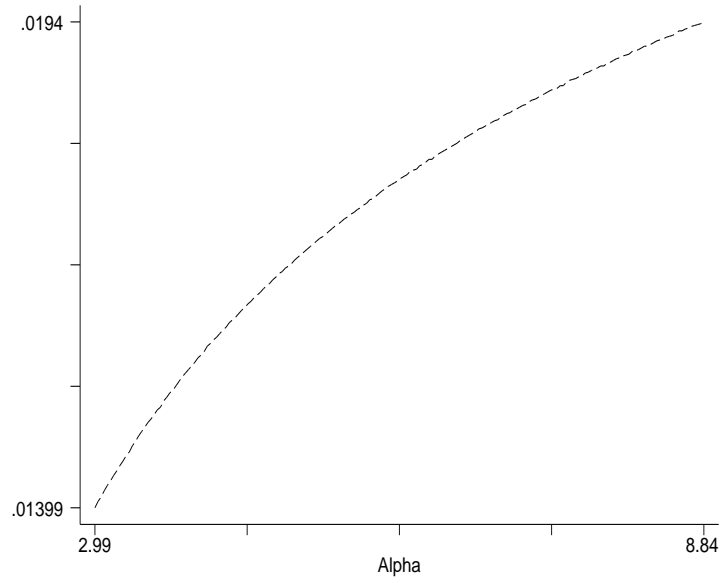


Figure 18: Evolution of g with respect to α .
 $\beta = -0.01, \rho = 0.02$ and $64.6 < E < 145, 109 < A_{\max} < 217$

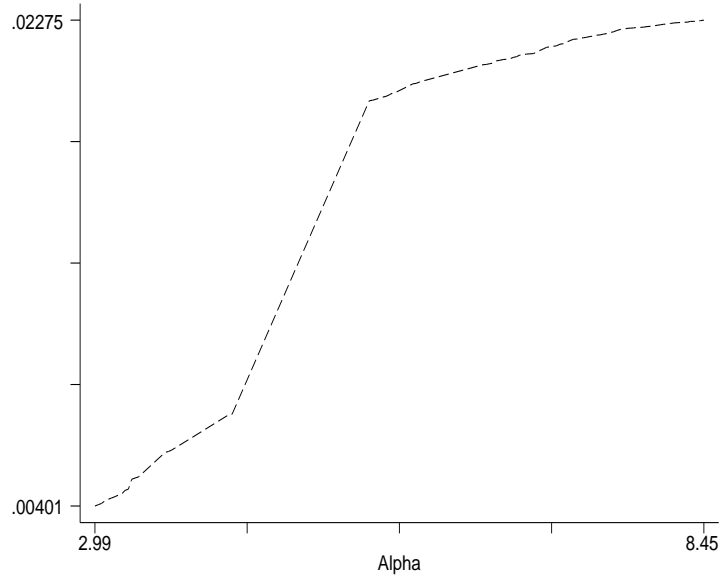


Figure 19: Evolution of g with respect to α .
 $\beta = -0.017, \rho = 0.02$ and $38 < E < 85, 64 < A_{\max} < 125$

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