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Optimal health investment with separable and non-separable preferences*

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Abstract

We use a general equilibrium framework to study optimal health investment in a dynamic model where agents derive utility from consumption and health. The steady state and the dynamics of the model are studied under separable and non-separable preferences. A shock undermining health which increases health expenditure and weakens the income base, not only affects savings but also compromises the consumption capacity. The magnitude of the effects strongly depends on the preferences. The dynamics of the model includes the equilibrium dynamics and bifurcations. Simulation experiments lend additional supports to our findings in favor of the non-separable preferences.

JEL classification: C61; C62; I15; E21
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1 Introduction

One of the most challenging tasks policy makers are facing in both developed and developing countries is the design of an 'efficient' health care system to improve population welfare. In the US for instance, this unachieved goal is commonly acknowledged to be as high a priority as that of Administration. In Europe where health care systems have already been put to the test, inefficiency usually leads governments to regularly introduce a variety of new schemes (generalisation of generic drugs, dropped from the market of inefficient drugs, obligation for people to have a family doctor, etc.) in order to reduce health spending and to improve the health care system. In the developing world, the economic consequences of diseases like AIDS, Malaria, etc. (sharp decline in life expectancy, fall in productivity, etc.) reinforced the idea of formulating a well adapted health care system without large long-run costs in terms of income per capita as recently pointed out by Acemoglu and Johnson (2007).[1] Understanding the economic mechanisms that drive the design and implementation of such policies is a key issue. The aim of this study is to highlight some of these economic mechanisms, mainly those related to consumption, savings and the financing of health expenditure in a context of health impairment.

Improvement of health is an important social and economic objective, which has obvious direct returns in the sense it favors longer and better life, but also a large indirect effect through the acceleration of economic growth.[2] In fact, the interplay between economic growth and health has been and still is a source of important debate in the literature since evidently, health is a human capital. In an illuminating assessment of this question, Sen (1998) pointed out the crucial connection between mortality and growth, and highlighted the ability of countries to reduce mortality as a test of their economic performance. Sen (1998) also outlined that the forces that contribute to an increase or a reduction of mortality often have economic causes.[3] Moreover, the increase of health expenditures in both developed and developing countries calls for a debate on health policies focusing on limiting the growth of health spending as a rational response to changing economic conditions notably the growth of income per capita as life expectancy increases. The role of health expenditure in improving longevity was recently studied by Hall and Jones (2007) who showed that spending on health to improve longevity led individuals to procure additional periods of utility, and that the marginal utility of life extension does not decline.[4] Wastson (2006) studied to which extent differential levels of investment in public health inputs explain observed differences in health outcomes across socioeconomic and racial groups. Bhalotra and Rawlings (2011) investigate the intergenerational persistence of health across time and region as well as across the distribution of maternal health. The structure of utility functions was empirically tested by Evans and Viscusi (1991). The authors used a survey data on risk-dollar tradeoffs for minor health effects and showed that health impact does not alter the structure of the utility function in a fundamental way. This paper contributes to the current debate by studying the effect of health shocks and the financing of health expenditure on consumption and saving, while elaborating in depth on the role of agents's preferences structure in that respect.

In Grossman's (1972) standard model, health is considered as a capital stock that increases

[1] Over the last few centuries, it has been recognised that improvements in health care are the main driving force of life expectancy in both developed and developing countries (Kirkwood, 2008).

[2] See e.g., Chalkley and Malcomson (1998) and Deaton (2003) for a nice contribution to the debate.

[3] See also Cervellatti and Sunde (2005, 2007) and de la Croix (2008).

[4] See the recent study of Dolan and Kahnemann (2008) on interpretations of utility in the context of non-market goods such as health. See also Bleichrodt and Pinto (2005) for some applications of non-expected utility in the health area.

with investment, by buying into health services, medical goods, or spending time on health related activities. However, it also decreases naturally through the ageing process. As pointed out by Gjerde et al. (2001), there are three key reasons why people want to improve health:

i) Since health directly enters the individual's utility, it can be considered as a consumption commodity.

ii) Health determines the total amount of time available for monetary and non-monetary activities. Therefore it is an investment commodity.

iii) Being healthy lengthens the life span and lessens the likelihood of premature mortality. Thus health is a determinant factor of longevity.

One key feature in the traditional modeling strategy *à la* Grossman is that preferences are separable in health and ordinary consumption. As a result, the returns from these two goods are independent. However, recent evidence stressed the fact that ordinary consumption is also crucial for health. According to the *World Health Organization* 2002 report, while overweight status and obesity coexist with stunting and micronutrient malnutrition, undernutrition has long been considered to represent both a consequence and cause of poor human health, underdevelopment, and underachievement throughout life.[5] Undernutrition with respect to energy, protein, etc. can adversely affect the quality of life, impair resistance to infection and diseases, and decrease span of life.

Following Grossman (1972)'s consumption model, Dardanoni and Wagstaff (1990) introduce the uncertainty within a static version of the same setup and find greater uncertainty results in an increase in the demand for medical care. However, a static framework does not allow for adjustments over time and may not fully characterise the effects of uncertainty. This issue was challenged by Picone et al. (1998) who analyze the effect of uncertainty of the incidence of illness on the precautionary behavior of elderly individuals within a stochastic dynamic model à la Grossman. The approach of Picone et al. (1998) is interesting in that the optimization problem is a result of a series of sequential decisions rather than one large maximization. The sequential decisions elaborate on the fact that an individuals current demand for health partly depends on previous decisions and on his/her uncertain expectations of the future. The authors found that more risk averse individuals will exhibit extra precautionary behavior, especially in the form of additional medical expenditures. Moreover, very risk averse individuals may respond to an increase in uncertainty by reducing their savings. This follows from the fact that increases in medical care are not completely offset by reductions in consumption. However, the optimization problem of Picone et al. (1998) does not have a closed form solution. As a result, the standard comparative statics cannot analytically be derived and the optimal paths of consumption and medical care expenditures were numerically implemented.

Ehrlich and Chuma (1990) specified a demand function for quantity and quality of life. Their model is based on Grossman (1972)'s type specification using the consumption-investment commodity aspect of health. The authors calculated that optimal health and longevity are correlated to the endowment of wealth, rather than necessarily current income. Gjerde et al. (2001) analyzed the impact of adaptating to a falling health state on the demand for health and medical care by integrating adaptation processes in the pure consumption model of Grossman (1972). They also introduced the uncertainty of longevity and their simulation experiments showed that adaptation affects health by lowering the incentives to invest in health, as well as smoothening the optimal health stock path over life cycle.

[5] See e.g. the recent study of Nonnemaker et al. (2008).

The contribution of Hazan and Zoabi (2006) is of particular interest as the authors challenge the conventional view on the relation between longevity and growth by arguing that greater longevity may have contributed less than previously thought for the significant accumulation of human capital during the transition from stagnation to growth. The leading mechanism of such finding is that in contrast to longevity, improvements in health are more likely to generate quantity quality tradeoff from parents behavior of their offspring. Greater longevity positively affects not only the returns to quality but also the returns to quantity. The result of Hazan and Zoabi (2006) relies on the homothetic preferences of parents with respect to the quantity and quality of their children. As a result, in a framework in which longevity is neutral, health can induce quantity-quality tradeoff. This points out the key role of preferences in modeling the relation between health and economic indicators such as consumption, saving, growth, etc. In this paper, we address this issue by studying a broad mechanism of separable and non-separable preferences in consumption and health.

At a first glance, one may observe that consumption and health are interconnected but we cannot make conclusions about the net effects on health investment. Hence, it is very difficult, at least analytically, to slice on the net effect of a high health deterioration rate and low health productivity on health investment. It is important to note that not all diseases have the same effects on individuals. For instance, we can distinguish between virulent epidemics which can kill in a very short period of time, and strongly affect an individual's health capital and life expectancy on the one hand and epidemics which do not kill in the short term, but confine people to bed, or constitute a handicap for the rest of their life on the other.[6] Therefore, in order to simplify and to avoid potential ambiguities in our modelling which entails the analysis of all factors, including health shocks that can undermine health capital, we have two different health shocks: one that accelerates the health deterioration rate, and another that decreases the efficiency of health investment.

One of the main economic implications of health shock is a probable and significant distortion in saving behavior. For example Freire (2002) studied the impact of AIDS on household savings in South Africa. The author found that the pandemic, and the associated sharply declining life expectancy is likely to shift savings downward for a while, therefore restricting economic growth and standard of living in the medium and long-run. Chakraborty (2004) considered the problem of public investment in health within the framework of overlapping generation models. The author showed that in poor countries where life expectancy is weak, individuals are more likely to discount the future and thus less inclined to save. Cuddington and Hancock (1994) also stated that, health expenditure induces a decrease in savings at the expense of capital accumulation. However, this is questionable since health expenditure is harmful to consumption. Therefore, there is an overriding issue as to how to deal with savings in the context of health depreciation.[7]

Our study contributes to the literature in several aspects. Firstly, we adopt a more general

[6]Among those diseases we can quote Plague, Spanish Influenza, Cholera, Meningitis, Virus of Marburg (discovered in Marburg in the north of Germany in 1967). The second kind of epidemic can be assimilated to endemic diseases. The effect of these diseases is that, many people die but many people survive but are reduced to poverty because of the loss of productivity, and their lives are blighted by frequent bouts of illness. Among them we find AIDS, paludism, Leishmaniasis (the second-largest parasitic killer in the world after malaria).

[7]Looking at this from a different perspective, Acharya and Balvers (2004) revisited the concept of intertemporal preferences replacing the unobservable concept of utility function by an observable health function. They showed that the rate of time preference varies in an intuitive way with changes in conditional lifetime, initial wealth age, and the marginal productivity of consumption in affecting health. Dusansky and Koç (2006) emphasised the role of uncertainty examining the interconnection between individual consumer's demand for medical care and choice of health insurance coverage (see also Cameron et al., 1988 and Dardanoni and Wagstaff, 1990). It is worthwhile to mention that those aspects of the issue are out of the scope of this study.

set-up by considering both separable and non-separable preferences in consumption and health, meaning that in the latter case consumption is also crucial for health. Secondly, we investigates the effects of health shocks on the subsequent life cycle, in particular on savings and consumption in a general equilibrium setting. In order to have a better picture of the life cycle aspect of the issue, we also include a final good sector. To the best of our knowledge this aspect was not studied so far as most studies rely on partial equilibrium demand model for health. Thirdly, even if our framework is much more complex, we are able to fully characterise the analytical solutions of the optimization problem and to study the dynamics of the model. The latter includes the dynamics of transition and the characterization of *bifurcations*. We shall see that the picture is quite sophisticated, depending on the assumed preferences. We pay a particular attention to the way health enters the utility function, and focus on the effects of health depreciation on savings, investment in health capital and consumption. Our framework postulates that the marginal utility of consumption depends on health capital and vice-versa. For simplification, we consider that the lifetime of individuals is infinite, thus removing any uncertainty on that side. However, this facet of our approach is closely related to the framework of Grossman (1972) and Ehrlich and Chuma (1990) in that health capital is still a determinant of lifetime utility. Moreover, by using a *constant-relative-risk-aversion* utility, we allow households to shift consumption between different periods according to their degree of risk aversion. In this context, we study agents' behavior facing health depreciation (which can be interpreted for instance as resulting from epidemics) which either accelerates the depreciation of health capital or decreases the productivity or efficiency of investment in health.

The chief outcomes of this paper demonstrate that a shock undermining health (increasing health expenditure and weakening the income base), not only affects savings, but also compromises the consumption capacity as well as the human and physical capital of the economy undercutting the process of economic development. We show that the magnitude of these effects depends on the assumed preferences. It is worthwhile noticing that our interest is to look at the steady states and the dynamics as well. The analysis is therefore relevant for permanent or long run changes in the depreciation rates of health. From the dynamics perspective, a nice picture that emerges from the transitional dynamics is the geometrical illustration of the common parameters support that allows us to link the two polar preferences (separable and non-separable). Additionally, we show that *bifurcations* emerge with separable preferences while non-separable preferences do not present such picture. We also provide simulation experiments of this dynamics to study the role of preferences. We argue that the model with separable preferences gives implausible predictions whereas the model with non-separable preferences is likely to explain important dimensions of empirical facts.

While emphasising the *preferences* aspects of the issue, it is not our intention to deny the relevance of other aspects such as the effects of health investments on expected lifetime, the role of preventive health care, etc. We believe that a thorough study of the mechanisms implied by preferences deserves a special attention while keeping other aspects for future investigation. Moreover, it is worthwhile to notice that health status is a broad concept that goes beyond the mere presence or absence of diseases.[8] In our case, the term health status refers to the general health situation of the individual.[9]

The remainder of the paper proceeds as follows. Section 2 provides empirical background for

[8]See e.g., Goenka and Liu (2010), Ryan and Vaithianathan (2003), Boucekkine and Laffargue (2010) and d'Albis and Augeraud-Véron (2008).

[9]Indeed, the way people report their health status is directly related to their use of medical services, which include visits to doctors or dentists, hospital stays, medicinal prescription, etc.

further motivations to our framework. Section 3 studies the model with separable preferences meaning that health and consumption enter additively into the utility function. Section 4 is devoted to the model with non-separable preferences where there is multiplicative interaction between health and consumption. Section 5 develops the equilibrium dynamics of the model under both preferences assumption. Section 6 analyzes the emergence of bifurcations and provides simulation experiments to support the findings. Section 7 summarises and discusses our findings with prior contributions.

2 A brief review of empirical facts

In this section, we briefly put forward some specific aspects of health which may help further motivating our approach. Health impairment and malnutrition often occur in tandem. Indeed, poor nutrition increases the risk and progression of disease and in turn, disease exacerbates malnutrition. Curtis (2004) outlined a possible relationship between nutrition and scarlet fever. The author suggested that poor nutrition during pregnancy may have caused women to give birth to children who were particularly susceptible to scarlet fever.[10] Bobat et al. (1997) studied prenatal HIV-infected children in Italy. The authors concluded that breastfeeding not only protects infants from common childhood illnesses, but it also could slow down the progression to AIDS for HIV-infected children. However, the advantages of breastfeeding were lost by the time children reached five years of age.[11]

Many factors may contribute to disease evolution making the identification of single causes very difficult. However, the observations above suggest that improved nutrition and the prompt treatment of infections in HIV infected individuals may delay the onset of AIDS. There is evidence from several studies that macronutrients play a role in HIV disease progression. Friis and Michaelsen (1998) focused on HIV-infected men in the USA. They suggested that high intakes of riboflavin, vitamin E and iron, and possibly vitamin A, C and thiamin were associated with reduced disease progression. Moreover, Fawzi et al. (1998) produced data from a randomised, controlled vitamin A trial among pre-school children with acute pneumonia in Tanzania. They concluded that vitamin A prolonged the life expectancy of the HIV infected, suggesting that vitamin A may play a role in slowing the course of HIV infection in children.

VanMaanen (1988) led two exploratory studies with elderly US and UK people in various age-groups. He found that the perception of health, by the self proclaimed 'healthy' elderly American was more 'a state of mind'. Whereas the 'un-healthy' elderly British person interpreted it as 'the state of absence of disease'. In each case, health maintenance behavior patterns valued were: balanced nutrition and physical exercise, etc. Subsequently, good nutrition throughout adult life will help protect against diseases such as diabetes, coronary heart problems, strokes and some cancers. For instance, the March 2003 report of the East Midlands Regional Assembly stated that, healthy eating could lead to a 20% reduction in deaths from chronic diseases, and appropriate dietary advice can prevent physical and mental deterioration, and improve the quality of life of older people.

Studies on human capital allow us to analyze the effects of nutrition on labor productivity. Fogel (1997) stated that nutritional improvements contributed to between 20 and 30% of the income per capita growth in England during 1780-1979. The functionality of the individual and his capacity to work in a productive way depended partly on her/his nutrition. In the same way,

[10] See e.g. Blaum (2005) for an empirical assessment of the role of pregnancy employment on health at birth.

[11] See also Boucekkine et al. (2008), Geoffard and Philipson (1996,1997), Gersovitz and Hammer (2004,2005), Kremer (1996), and Tanaka (2005) for a study of the interplay between parental leave and child health.

Seshadri and Tara (1989) showed that in India, iron deficiency in children affects their cognitive capacity and their performance at school. Therefore, consumption of un-healthy goods plays an important role when addressing health.

There are several channels through which diseases can affect the economy. Some recent contribution have advocated the human capital channel. For example, Corrigan et al. (2005) have used a calibrated OLG model to analyze the effect of the drop in life expectancy on investment. The authors found a significant increase in the number of orphans as a result of AIDS. McDonald and Roberts (2006) used an econometric model combining growth and health capital equations.[12] Applied on African countries, the model predicted the substantial effects of the epidemic: the marginal impact on income per capita of a one percent increase in HIV prevalence rate is minus 0,59%. The authors concluded that while the human and social costs of the HIV/AIDS are major causes for concern, the macroeconomic effects of the epidemic are by no way negligible. Thornton (2008) evaluated an experiment in which individuals in rural Malawi were randomly assigned monetary incentives to learn their HIV results after being tested.

In a highly controversial paper, Young (2005) claimed that AIDS severely lowers fertility for two main reasons: On the one hand, the epidemic has undoubtedly reduced willingness to engage in unprotected sexual activity. On the other, the high mortality of adult males and the resulting scarcity of labor are likely to increase the value a woman's availability. Both channels are arguably strong enough to induce a long-lasting decrease in fertility, which may cause future consumption per capita to rise. Using a Barro-Becker based empirical model, the author found that in the case of South-Africa, this decreasing fertility engine is so strong that it dominates the human capital channel put forward by Corrigan et al. (2005). Hence, AIDS might well be interpreted as a 'gift of the dying' for future South African generations. Nonetheless, such a finding has been challenged by some authors, including the very interesting paper of Kalemli-Ozcan (2006) who studied the fertility issues on a panel of 44 African countries between 1985-2000. The author showed that the HIV/AIDS affects the total fertility rate positively and the school enrollment rates negatively.[13]

Several studies have elaborated on the relevance of the interdependence between health investment, life expectancy and economic growth as well as outlining the crucial role of public policies (Watson, 2006). In this respect, country historical figures are very interesting to put forward. As outlined by Jones (2001), in 1998, the U.S. spent 13.% of its GDP on goods and services related to health care which represents a big increase compared to the 5.1% spending in 1960. The 1998 spending shares were 10.6% in Germany, 9.6% in France, 9.5% in Canada, 7.4% in Japan, and 6.7% in the U.K. At the same time, the pattern of life expectancy increase is the opposite, considering that U.S. life expectancy at birth was 57.1 years in 1929, 68.2 years in 1950 and up to 75.5 years by 1990.

Shaw et al. (2005) studied life expectancy production function for a sample of OECD countries using a parametric specification. The authors found that pharmaceutical consumption has positive impact on life expectancy for middle age and older. Shaw et al. (2005) also noted that this relation is sensitive to age distribution. Peltzman (1987) studied the effect of life expectancy at birth on wealth and government health spending and found only health to be a significant determinant. Miller and Frech (2000) used OECD age strata data and found that the determinants of life expectancy in each stratum regression were wealth, pharmaceutical and non-pharmaceutical medical expenditures.

[12] See also Bloom and Mahal (1997) for a very careful econometric study on the effect of epidemics on economic growth.

[13] See also Zivin et al. (2009).

The literature also provides strong evidence support as to how healthiness is closely connected to economic development. Gallup and Sachs (2001) argued that wiping out malaria in sub-Saharan Africa could increase the continent's per capita growth rate by as much as 2.6% per year. In the U.S. between 1980 and 2000, the annual number of deaths fell by 16%, life expectancy increased across all age groups by an average of 5%, and the number of hospital days fell by 56%.[14] Additionally, using the value of a statistical life method[15], the Medical Technology Assessment and Policy (MEDTAP) revealed in their 2002 report that, annual health care expenditure per person increased by $2,254 in the U.S. between 1980 and 2000. This report also stated that for every additional $1 spent on health care, the value of health gains ranged from $2.40 to $3.00, and at least a 40% increase in life expectancy is directly attributable to additional health care. However, as mentioned by Hall and Jones (2007) how much of this increase is exactly due to increased health spending is unclear, but the large gains in life expectancy clearly represent one of the major accomplishments of the 20th century. One way to explain for this increase in share of health expenditure should be to consider health as a superior good. As people get richer, consumption rises but they devote an increasing share of resources to health care.[16]

Hereafter, we study the effects on subsequent life cycle of long-lived shocks undermining health with either an acceleration of health capital deterioration, or a decrease in health investment efficiency.

3 The model with additively separable preferences

Most studies investigating the interaction between health and the ways it affects utility use an additive structure. As a result health status and consumption are additively separable in the utility function implying that the marginal utility of consumption is independent from health status. Our first approach consists in exploring this framework as a benchmark model. However, in contrast to previous studies that used additive models, we add a final good sector to better understand the life cycle aspect of the issue. Additionally we also build upon the literature with separable preferences by investigating the equilibrium dynamics and bifurcations. In the next section, we will relax the separability assumption.

3.1 The consumer's problem

We assume (like Hall and Jones, 2007) that health status and consumption at time z are additively separable in utility. Then the agent maximises the lifetime utility $V : \mathbb{R}^2_+ \to \mathbb{R}$:

$$\int_z^\infty V(C(z), M(z))e^{-\rho z}dz = \int_z^\infty \left[U(C(z)) + \varphi(M(z))\right] e^{-\rho z}dz \tag{1}$$

where $V(C, M)$ is the instantaneous utility derived from consumption goods $C(z)$, and stock of health capital $M(z)$, and ρ is the time preference or discount rate. We assume that $\varphi(M(z))$

[14] However, as noticed by Bhargava et al. (2001), life expectancy in a country is a broad measure of population health, though it needs not accurately reflect the productivity of the labor force.

[15] The value of life is the cost of reducing the (average) number of deaths by one. When deciding on the appropriate level of health care spending, a typical method is to equate the marginal cost of the health care to the marginal benefits received. In order to obtain a marginal benefit amount, some estimation of the dollar value of life is required.

[16] Newhouse (1992) has pointed out the crucial role played by the development of new technologies in the rise of health spending. In fact, the invention of new and expensive medical technologies causes health spending to rise over time.

is an strictly increasing and concave function in M, and can be considered as the amount of healthy time. The law of motion of non-human assets and health are respectively given by

$$\dot{A}(z) = r(z)A(z) + w(z) - C(z) - m(z) \tag{2}$$

$$\dot{M}(z) = \psi(m(z)) - \delta_M M(z) \tag{3}$$

where $r(z)$ is the interest rate. We assume that each individual supplies one unit of work per unit of time, then $w(z)$ is the wage rate, $m(z)$ is the flow of gross investment in stock of health capital $M(z)$. The health investment $m(z)$ is produced with a decreasing-returns-to-scale technology via the function $\psi(m(z))$.

Consistent with Ehrlich and Chuma (1990), we consider that the stock of health capital can be maintained or increased through purposive investments $m(z)$. However, health is submitted to a natural biological deterioration at the rate δ_M. Thus, in contrary to Ehrlich and Chuma (1990), we assume a constant rate of health depreciation. However, the greater the health that one intends to maintain in later years, the earlier one must initiate significant investments in counteracting the depreciation of health.

Individuals maximise lifetime utility (1), subject to the state variables $\dot{A}(z)$ and $\dot{M}(z)$ in equations (2) and (3) respectively. The Hamiltonian and optimality conditions of this optimal control problem are

$$\begin{aligned} \mathcal{H} &= [U(C(z)) + \varphi(M(z))]\, e^{-\rho z} \\ &+ \lambda_A e^{-\rho z} [r(z)A(z) + w(z) - C(z) - m(z)] \\ &+ \lambda_M e^{-\rho z} [\psi(m(z)) - \delta_M M(z)] \end{aligned}$$

The first order conditions associated to this problem are given as:

$$\frac{\partial \mathcal{H}}{\partial C(z)} = U_C e^{-\rho z} - \lambda_A e^{-\rho z} = 0 \tag{4}$$

$$\frac{\partial \mathcal{H}}{\partial m(z)} = \lambda_M \psi'(m(z)) e^{-\rho z} - \lambda_A e^{-\rho z} = 0 \tag{5}$$

$$\frac{\partial \mathcal{H}}{\partial M(z)} = [\dot{\lambda}_M - \rho\lambda_M] e^{-\rho z} = \lambda_M \delta_M e^{-\rho z} - \varphi'(M(z)) e^{-\rho z} \tag{6}$$

$$\frac{\partial \mathcal{H}}{\partial A(z)} = e^{-\rho z}[\rho\lambda_A - \dot{\lambda}_A] = r(z)\lambda_A e^{-\rho z} \Longrightarrow \dot{\lambda}_A = \lambda_A(\rho - r(z)) \tag{7}$$

The additional transversality conditions are:

$$\lim_{z\to\infty} \lambda_A(z) e^{-\rho z} A(z) = 0 \tag{8}$$

$$\lim_{z\to\infty} \lambda_M(z) e^{-\rho z} M(z) = 0 \tag{9}$$

We specify constant-relative-risk-aversion (CRRA) functions for $U(C(z))$ and $\varphi(M(z))$, and a decreasing return in health investment for function $\psi(m(z))$ as:

$$U(C(z)) = \frac{(C(z))^{1-\sigma_1}}{1-\sigma_1} \tag{10}$$

$$\varphi(M(z)) = b\frac{(M(z))^{1-\sigma_2}}{1-\sigma_2} \tag{11}$$

$$\psi(m(z)) = \pi(m(z))^{\alpha} \tag{12}$$

with $\sigma_1 > 0$, $\sigma_2 > 0$, $b > 0$ and $0 < \alpha < 1$. Here σ_1 is the inverse of elasticity of substitution between consumption at any two points in time, and σ_2 denotes the same for health capital. $U(C(z))$ and $\varphi(M(z))$ are strictly increasing and concave respectively in $C(z)$ and $M(z)$.

$\psi(m(z))$ represents the health investments function, which is concave in $m(z)$, reflecting the assumed diminishing returns in health investment. π is the productivity or efficiency of health investment. Increased health care productivity not only shifts the health production function upward, but causes each unit of health care to have a larger contribution to health as well. From Eq. (5) we can derive the marginal value of health capital relative to the marginal value of ordinary consumption, that is

$$\frac{\lambda_M}{\lambda_A} = \frac{1}{\psi'(m(z))} = \frac{1}{\pi\alpha(m(z))^{\alpha-1}} \tag{13}$$

Ehrlich and Chuma (1990) also assumed that the consumer is choosing death when his stock of capital $M(z)$ is under a certain minimal level $M_{\min}$. In our setup, we assume $M_{\min} = 0$. By doing so, we end up with a standard infinite time model, provided $M(z) \geq 0, \forall z$. This simplification will allow us to tackle much more comfortably the sophisticated optimization problem. Equations (4) and (7) yield the traditional Euler relation:

$$\begin{aligned} \dot{C} &= \frac{U_C(\rho - r)}{U'_C} \\ \Rightarrow \quad \frac{\dot{C}}{C} &= \frac{r-\rho}{\sigma_1} \end{aligned} \tag{14}$$

From (5) and (6) we obtain:

$$\begin{aligned} \frac{\dot{m}}{m} &= \frac{\delta_M + r}{1-\alpha} - \frac{\varphi'(M)\psi'(m)}{(1-\alpha)U_C} \\ &= \frac{\delta_M + r}{1-\alpha} - \frac{bM^{-\sigma_2}\pi\alpha m^{\alpha-1}}{(1-\alpha)C^{-\sigma_1}} \end{aligned} \tag{15}$$

The negative sign on the second term of the right hand-side of (15) implies that when the value of health increases, then health investment goes upward. Indeed, $\pi\alpha m^{\alpha-1}$ is the inverse of the unit value of health which lowers the whole term when this value increases, ensuring an increase in m. Let us now introduce the firm problem.

3.2 The firm's problem

We now proceed to the producer side. We consider a representative firm with neoclassical Cobb-Douglas technology:

$$F(K,L) = Y(z) = B(z)K(z)^{\epsilon}\left(L(z)\right)^{1-\epsilon}, \qquad 0 < \epsilon < 1$$

where $F(K,L)$ denotes the production function, $Y(z)$, $K(z)$ and $L(z)$ are respectively output, capital input and labor input respectively, while $B(z)$ is the technological level which grows at a constant rate. Let us denote $\hat{k} = \frac{K}{L}$ the capital-labor ratio. Then the output per labor can be rewritten:

$$f(\hat{k}) = \frac{Y(z)}{L(z)} = B(z)\hat{k}(z)^{\epsilon}$$

The maximization of the profit function under perfect competition allows to equalise the marginal cost of each factor with its marginal benefit. Therefore,

$$r(z) = \epsilon B(z)\hat{k}(z)^{\epsilon-1} - \delta \tag{16}$$

$$w(z) = f(\hat{k}(z)) - \hat{k}(z)f'(\hat{k}(z)) = (1-\epsilon)B(z)\hat{k}(z)^{\epsilon} \tag{17}$$

where $\delta \geq 0$ is the capital depreciation rate and w is the wage rate. Combining the demand and the supply sides, we can characterise the equilibrium of the economy as follows.

3.3 Equilibrium

In a closed economy with no public debt, aggregate financial wealth equals, by definition, the value of the capital stock. This implies that $A(z) = K(z)$ for all z. We can therefore write

$$\frac{\dot{\hat{k}}(z)}{\hat{k}(z)} = \frac{\dot{K}(z)}{K(z)} - n = \frac{F(K, L) - \delta K(z) - C(z) - m(z)}{K(z)} - n$$

and

$$\begin{aligned} \dot{\hat{k}}(z) &= f(\hat{k}(z)) - \hat{C}(z) - \hat{m}(z) - (\delta + n)\,\hat{k}(z) \\ &= B(z)\hat{k}(z)^{\epsilon} - \hat{C}(z) - \hat{m}(z) - (\delta + n)\,\hat{k}(z) \end{aligned} \tag{18}$$

where $\hat{C}(z)$ and $\hat{m}(z)$ are respectively the consumption and health expenditure per labor, and n (which for simplicity will be assumed null) is the population growth rate. Therefore, we are able to summarise the dynamics of the economy by the following non-trivial four dimensional system:

$$\begin{cases} \frac{\dot{\hat{C}}(z)}{\hat{C}(z)} = \frac{r(z)-\rho}{\sigma_1} \\ \frac{\dot{\hat{m}}(z)}{\hat{m}(z)} = \frac{\delta_M + r(z)}{1-\alpha} - \frac{bM(z)^{-\sigma_2}\pi\alpha m(z)^{\alpha-1}}{(1-\alpha)C(z)^{-\sigma_1}} \\ \dot{\hat{M}}(z) = \pi(\hat{m}(z))^{\alpha} - \delta_M \hat{M}(z) \\ \dot{\hat{k}}(z) = B(z)\hat{k}(z)^{\epsilon} - \hat{C}(z) - \hat{m}(z) - \delta\hat{k}(z) \end{cases} \tag{19}$$

with $\hat{k}(0)$ and $\hat{M}(0)$ given, plus the transversality conditions. The steady-state values of $\hat{C}$, $\hat{m}$, $\hat{M}$, and $\hat{k}$ are obtained by equalising $\dot{\hat{C}}$, $\dot{\hat{m}}$, $\dot{\hat{M}}$, $\dot{\hat{k}}$ to zero. We obtain:

$$\hat{C} = B\hat{k}^{\epsilon} - \hat{m} - \delta\hat{k} \tag{20}$$

$$\hat{m}^{\frac{\alpha}{\sigma_2}+1-\alpha} = \left(\frac{\delta_M}{\pi}\right)^{\frac{1}{\sigma_2}} \frac{b\pi\alpha(B\hat{k}^{\epsilon} - \hat{m} - \delta\hat{k})^{\sigma_1}}{\delta_M + r} \tag{21}$$

$$\hat{M} = \frac{\pi\hat{m}^{\alpha}}{\delta_M} \tag{22}$$

$$\hat{k} = B^{\frac{1}{1-\epsilon}}\left(\frac{\delta+\rho}{\epsilon}\right)^{\frac{1}{\epsilon-1}} \tag{23}$$

The following proposition characterises the solution of the system and the positivity of $\hat{m}$ and $\hat{C}$.

Proposition 1 *There is a unique solution $\hat{m}$ to the Equation (21), with $\hat{m} < B\hat{k}^{\epsilon} - \delta\hat{k}$.*

Proof. The proof is quite intuitive. Indeed, the left hand side of the equation (21) is strictly increasing in $\hat{m}$, while the right hand side is strictly decreasing, and the latter is equal to zero when $\hat{m} = B\hat{k}^{\epsilon} - \delta\hat{k}$ and equal to $(\frac{\delta_M}{\pi b})^{\frac{1}{\sigma_2}} \frac{b\pi\alpha(B\hat{k}^{\epsilon} - \delta\hat{k})^{\sigma_1}}{\delta_M + r}$ if $\hat{m} = 0$. Hence, there is a unique solution $\hat{m}$ to equation (21), with $\hat{m} < B\hat{k}^{\epsilon} - \delta\hat{k}$ which therefore means that $\hat{m}$ and $\hat{C}$ are always positive. $\square$

3.4 Comparative statics

In this section we study the static comparative of the model. We consider the effect of modifications in the parameters of the endogenous variables along the balanced growth path. We will consider two cases: the case where B is independent of the determinants of health capital. Then we examine an ad-hoc case where B is dependent on such determinants.

3.4.1 B independent of health capital

The following proposition characterises the behavior of health investment and consumption with respect to health depreciation rate.

Proposition 2 *Health investment (resp. consumption) is a strictly increasing (resp. decreasing) function of the health depreciation rate δ_M if and only if $\sigma_2 > \frac{\delta_M}{\delta_M+\rho}$. In contrast, savings are unaltered by changes in δ_M.*

Proof. See Appendix. □

Two comments are in order with respect to the above result. Firstly, there is a minimum value from which the marginal utility with respect to health capital (and thus to health investment) will drop if $\hat{M}$ or $\hat{m}$ is raised in response to the deterioration of health depreciation. Henceforth, high values of σ_2 are compatible with increasing health investments. In contrast, large values of σ_2 are likely to make the marginal welfare cost of such investment prohibitive. Secondly, the larger δ_M the lower the incentives to invest in health compared to investment in physical capital (which depreciation is kept constant). Indeed as δ_M rises, the lower bound of the values of σ_2 needed to increase health investment tends to unity. Now notice that the marginal rate of investment in physical capital, that is $B\epsilon\hat{k}^{\epsilon-1} - \delta$ is unaffected by changes in δ_M. This is due to the fact that when health depreciation do not affect total factor productivity (B) in our model, then gross investment $\hat{i}$ is equal to $\delta\hat{k}$ which is unrelated to δ_M. Therefore, in the steady state, savings do not get modified by changes in δ_M. It follows that when $\sigma_2 > \frac{\delta_M}{\delta_M+\rho}$, the rise in health investment in response to health depreciation is entirely paid by a decrease in consumption, which is at odds with Cuddington and Hancok's (1994) working assumption.

Proposition 3 *When health productivity π decreases, an increase in health investment occurs if and only if $\sigma_2 < 1$, while consumption decreases and savings remain unchanged. If $\sigma_2 = 1$, there is a neutral effect on the economic variables, and if $\sigma_2 > 1$ health expenditure decreases.*

Proof. See Appendix. □

A few comments are in order here regarding the interpretation of the model. A decrease in π has two effects on health investment $\hat{m}$. On one hand, it decreases the efficiency of health investment as featured in equation (22), therefore causing $\hat{m}$ to drop. On the other hand, a lower π increases the marginal value of health capital relative to the marginal value of the consumption good as one can notice in Equation (13), inducing an incentive to invest more in health capital. However, since a marginal drop in $\hat{m}$ is valued by the utility term $M^{-\sigma_2}$, the negative effect will dominate the positive one for σ_2 high enough, here $\sigma_2 > 1$. If $\sigma_2 = 1$, there is just a compensation between the two effects which yields a neutral impact. For low values of σ_2 allowing investment in health, the latter is always financed by a reduction in ordinary consumption, while savings remain constant. An important finding from this model is that faced with health depreciation, whether people reduce or increase their health expenditure, their saving remains constant, and an increase in health investment is moderately financed by a reduction of their consumption.

3.4.2 The case of productivity-decreasing health shock

We now suppose that the productivity B depends on the health productivity π and the health depreciation rate δ_M in an ad-hoc way. We will assume (like Cuddington and Hancok, 1994) that

a decreasing health capital induces lower productivity. More precisely, we will assume that all factors pushing down this capital has a negative effect on B. Therefore, we make the hypothesis that the health productivity π positively affects B, while the latter is negatively affected by δ_M. The following result holds.

Proposition 4 *Health investment (resp. consumption) increases (resp. decreases) in response to the health shock δ_M if and only if*

$$\sigma_2 < \frac{1}{1 - B_{\delta_M}\sigma_1\hat{k}^{\epsilon}(\delta_M + r)(B\hat{k}^{\epsilon} - \hat{m} - \delta\hat{k})^{-1}} + \frac{\rho}{\delta_M - \delta_M B_{\delta_M}\sigma_1\hat{k}^{\epsilon}(\delta_M + r)(B\hat{k}^{\epsilon} - \hat{m} - \delta\hat{k})^{-1}}$$

while savings always decrease.

Proof. See Appendix. □

This result is in line with the Proposition (2), however, it is worse in the case of health investment. Indeed, if δ_M increases infinitely, instead of $\sigma_2 > \frac{\delta_M}{\delta_M+\rho}$, we now need σ_2 in $]0;x[$ with $x = \frac{1}{1-B_{\delta_M}\sigma_1\hat{k}^{\epsilon}(\delta_M+r)(B\hat{k}^{\epsilon}-\hat{m}-\delta\hat{k})^{-1}} < 1$ to allow a rise in $\hat{m}$, where $B_{\delta_M} = \frac{\partial B}{\partial \delta_M} < 0$. It happens that the interval of evolution of σ_2 which would favor $\hat{m}$ is reduced by the additional resource constraint through the productivity B. On the ordinary consumption side, when it occurs, the increase in health expenditure is totally compensated by a fall in consumption (see proof in the Appendix). However, on the other side we observe a fall in the saving level, since the capital stock is negatively affected through the negative effect of δ_M on the productivity B. Compared to the case where the productivity B is not affected by the health parameters, at this point, to finance an increase in health expenditure, in addition to the fall in consumption, savings drop because of the loss of resources resulting from the loss of productivity. Furthermore, if the health shock has a large effect (i.e. $B_{\delta_M} \to -\infty$), health investment will never increase (the necessary and sufficient condition of proposition (4) tends to $\sigma_2 \leq 0$).

Proposition 5 *A downshift of the health productivity π is followed by an increase in health investment if and only if $\sigma_2 < \frac{1}{1+B_{\pi}\sigma_1\hat{k}^{\epsilon}\pi(B\hat{k}^{\epsilon}-\hat{m}-\delta\hat{k})^{-1}}$, meanwhile consumption and savings decrease.*

Proof. See Appendix. □

Let us comment on the meaning of this result. The interval in which the values of σ_2 are favorable to an increase in health expenditure is now reduced. Indeed we now need $\sigma_2 < \frac{1}{1+B_{\pi}\sigma_1\hat{k}^{\epsilon}\pi(B\hat{k}^{\epsilon}-\hat{m}-\delta\hat{k})^{-1}}$ instead of $\sigma_2 < 1$ to increase $\hat{m}$, where $B_{\pi} = \frac{\partial B}{\partial \pi} > 0$. In other words, from now on, even for $\sigma_2 \in \left]\frac{1}{1+B_{\pi}\sigma_1\hat{k}^{\epsilon}\pi(B\hat{k}^{\epsilon}-\hat{m}-\delta\hat{k})^{-1}};1\right]$ a fall in π is followed by a fall in $\hat{m}$. This phenomenon is certainly due to the fact that the agent has also a resource problem. Indeed a fall of π, has a negative effect on the wages through the productivity B, thus reducing his financial possibilities. In addition, given that there is also a negative impact on the gross investment in physical capital, we can expect a reduction of saving. As in Proposition (4), a drop in consumption and savings will make it possible to pay the increase in health expenditure.

To conclude this section, it is worth noting that while some studies like Cuddington and Hancok (1994) showed that an increase in health expenditure is inevitably accompanied by a decrease in savings, our study demonstrated that health expenditure can be fully financed by a decrease in saving and consumption, or by a reduction of consumption while saving remains unaltered. However, in this benchmark model, the determination of the relationship between

health investment (and therefore consumption and savings) and the health parameters δ_M and π entirely depends on the health elasticity σ_2. This constraint may be due to the fact that health and ordinary consumption enter utility in an additive way. Therefore, marginal utility of consumption is independent of health capital and vice-versa. We develop hereafter an alternative model where consumption is also a determinant factor of good health, as largely previously explained.

4 The model with non-separable preferences

We move from the above pattern, where consumption and health enter into the utility function in an additive way. Remember that in the later case the marginal utility of consumption is independent from health, and this does not fit in for instance with the notion that good nutrition is also important for health. Indeed, healthy eating might lead to a reduction in mortality from chronic diseases, and appropriate dietary advice can prevent physical and mental deterioration, and improve the quality of life. Evidence from Friis and Michaelsen (1998) supports this rationale. Therefore, consumption is also crucial for health. The alternative model we propose below seeks to account for this important aspect.

4.1 Model set-up

The individual's lifetime utility $V : \mathbb{R}^2_+ \to \mathbb{R}$ is now specified as follows:

$$\int_z^\infty V(C(z), M(z))e^{-\rho z}dz = \int_z^\infty U(C(z))\varphi(M(z))e^{-\rho z}dz \tag{24}$$

In order for the function $V(\cdot)$ to fulfil the standard property of positive marginal utility, we have to assume $\sigma_1 < 1$, $\sigma_2 < 1$. We disregard the case $\sigma_1 = \sigma_2 = 1$ because it imposes further constraints on the values of economic variables. Indeed, when $\sigma_1 = 1$, then $V_M = \ln(C)\varphi'(M)$, which requires $C > 1$. Therefore, we focus on the case $\sigma_1 < 1$, $\sigma_2 < 1$, which allows us to compare results with the analogue benchmark model.

Individuals seek to maximise lifetime utility (24), which depends on a consumption stream $C(z)$ subject to the state variables $\dot{A}(z)$ and $\dot{M}(z)$ in Equations (2) and (3) respectively. The hamiltonian of this problem is given by:

$$\begin{aligned}
\mathcal{H} &= U(C(z))\varphi(M(z))e^{-\rho z} \\
&+ \lambda_A e^{-\rho z}\left[r(z)A(z) + w(z) - C(z) - m(z)\right] \\
&+ \lambda_M e^{-\rho z}\left[\psi(m(z)) - \delta_M M(z)\right]
\end{aligned}$$

The optimality conditions are:

$$\frac{\partial \mathcal{H}}{\partial C(z)} = U_C\varphi(M(z))e^{-\rho z} - \lambda_A e^{-\rho z} = 0 \tag{25}$$

$$\frac{\partial \mathcal{H}}{\partial m(z)} = \lambda_M\psi'(m(z))e^{-\rho z} - \lambda_A e^{-\rho z} = 0 \tag{26}$$

$$\frac{\partial \mathcal{H}}{\partial M(z)} = [\dot{\lambda}_M - \rho\lambda_M]e^{-\rho z} = \lambda_M\delta_M e^{-\rho z} - U(C(z))\varphi'(M(z))e^{-\rho z} \tag{27}$$

$$\frac{\partial \mathcal{H}}{\partial A(z)} = e^{-\rho z}[\rho\lambda_A - \dot{\lambda}_A] = r(z)\lambda_A e^{-\rho z} \Longrightarrow \dot{\lambda}_A = \lambda_A(\rho - r(z)) \tag{28}$$

with the transversality conditions:

$$\lim_{z\to\infty} \lambda_A(z)e^{-\rho z}A(z) = 0 \tag{29}$$

$$\lim_{z\to\infty} \lambda_M(z)e^{-\rho z}M(z) = 0 \tag{30}$$

We use the same functional forms as in the benchmark set-up for $U(C(z))$, $\varphi(M(z))$ and $\psi(m(z))$. However, for $\varphi(M(z))$, we get rid of the parameter b which is unimportant in what follows. Equation (26) allows to find the shadow price $\frac{\lambda_M}{\lambda_A} = g(t) = \frac{1}{\psi'(m(z))}$ which is the unit value of health capital. Moreover, a continuous stock equilibrium condition for health can be derived from Equation (27):

$$\frac{\dot{\lambda}_M}{\lambda_A} = \frac{\lambda_M}{\lambda_A}(\rho + \delta_m) - \frac{U(C(z))\varphi'(M(z))}{\lambda_A} \tag{31}$$

Combining Equations (28) and (31), we obtain:

$$g\left(\delta_m + r - \tilde{g}\right) = \frac{1}{\lambda_A(0)} U(C(z))\varphi'(M(z)) e^{\int_0^z (r(z)-\rho)dz} \tag{32}$$

with $\tilde{g} = \frac{\dot{g}}{g}$ and $g = \frac{1}{\pi\alpha}(m(z))^{1-\alpha}$. Equation (32) states that the instantaneous user cost of health capital should equal the instantaneous marginal benefit. The explicit expression of the shadow price can be derived from Equation (32) as %

$$g = \bar{g} e^{\int_z^\infty (\delta_m + r(z))dz} + \frac{e^{\int_z^\infty (\delta_m + r(z))dz}}{\lambda_A(0)} \int_z^\infty \left[U(C(v))\varphi'(M(v)) e^{\int_0^v (r(z)-\rho)dz}\right] e^{-\int_v^\infty (\delta_m + r(s))dz} dv \tag{33}$$

where $\bar{g}$ is an integration constant, reflecting the limit value of g when z tends to infinity (assuming that such a limit value is finite). Equation (33) means that the value of health capital is determined by the asymptotic value of life extension (the first term of the right-hand side), and the value of healthy life or the discounted value of health benefits (second term).

We can now explore the connection between the path of health investment and the optimal consumption over life cycle. Using Equation (25), we obtain $\lambda_A = U_C \varphi(M(z))$ and

$$\frac{\dot{\lambda}_A}{\lambda_A} = \rho - r(z) = \frac{\dot{C}(z) U'_C}{U_C} + \frac{\dot{M}(z)\varphi'(M(z))}{\varphi(M(z))}$$

Then the path of the optimal consumption is:

$$\dot{C}(z) = -\frac{U_C}{U'_C}(r(z) - \rho) - \frac{U_C}{U'_C}\frac{\varphi'(M(z))}{\varphi(M(z))}\dot{M}(z) \tag{34}$$

The first term on the right-hand side is the slope of the consumption path times the difference between the rate of interest and time preference. The second term reflects the interaction between the individual's health and his capacity to consume. Next we replace U_C, U'_C and $\varphi'(M(z))$ by their respective expressions in (34) for the elasticity parameters to show up:

$$\begin{aligned} \frac{\dot{C}(z)}{C(z)} &= \frac{1}{\sigma_1}(r - \rho) + \frac{1-\sigma_2}{\sigma_1}\frac{\dot{M}(z)}{M(z)} \\ &= \frac{1}{\sigma_1}(r - \rho) + \frac{1-\sigma_2}{\sigma_1}\left(\frac{\psi(m)}{M(z)} - \delta_M\right) \end{aligned} \tag{35}$$

In the typical Ramsey setup, we have the following optimal rule:[17]

$$r = \rho - \frac{U'_C C}{U_C}\frac{\dot{C}}{C}$$

[17] The Keynes-Ramsey standard term known as the difference between the interest rate and the pure rate of time preference.

This means that the interest rate should cover the impatience rate (or rate of time preference) and the marginal utility loss as captured by the term $-\frac{U_C' C}{U_C}\frac{\dot{C}}{C}$, where $\frac{U_C' C}{U_C}$ is the elasticity of marginal utility with respect to consumption. Equation (34) can be rewritten as:

$$r = \rho - \frac{U_C' C}{U_C}\frac{\dot{C}}{C} - \frac{\varphi'(M(z))}{\varphi(M(z))}M(z)\frac{\dot{M}(z)}{M(z)}$$

In this model, the interest rate should also account for the possible loss or gain in health capital due to any marginal change in consumption and savings. This reflects the dependence between the (marginal) welfare impacts of consumption and health investment. We shall now extract the optimal law of motion for health investment m. Using Equation (26), we obtain:

$$\begin{aligned} \frac{\dot{\lambda}_M}{\lambda_M} &= \frac{\dot{\lambda}_A}{\lambda_A} - \frac{\dot{m}\psi''(m)}{\psi'(m)} \\ &= \rho - r(z) + (1-\alpha)\frac{\dot{m}(z)}{m(z)} \end{aligned} \tag{36}$$

From equation (27) we get

$$\begin{aligned} \frac{\dot{\lambda}_M}{\lambda_M} &= -\frac{U(C(z))\varphi'(M)\psi'(m)}{\lambda_A} + \delta_M + \rho \\ &= -\frac{U(C(z))\varphi'(M)\psi'(m)}{U_C\varphi(M)} + \delta_M + \rho \end{aligned} \tag{37}$$

Combining these two equations yields the path of the optimal health investment:

$$\frac{\dot{m}(z)}{m(z)} = \frac{r+\delta_M}{1-\alpha} - \frac{U(C(z))}{\varphi(M)}\frac{\varphi'(M)\psi'(m)}{(1-\alpha)U_C} \tag{38}$$

$$= -\frac{1-\sigma_2}{1-\sigma_1}\frac{\pi\alpha m^{\alpha-1}}{1-\alpha}\frac{C(z)}{M(z)} + \frac{r+\delta_M}{1-\alpha} \tag{39}$$

The next proposition states required concavity conditions for $U(C)$ and $\varphi(M)$.

Proposition 6 *The Mangasarian's (1966) sufficient conditions for our optimal control problem are fulfilled if and only if* $\frac{\sigma_1\sigma_2}{(1-\sigma_1)(1-\sigma_2)} > 1$.

Proof. See Appendix. □

Let $\mathbf{H} = [\mathcal{H}(C, M, m, A, r, w)]$ be the Hessian matrix of the Hamiltonian $\mathcal{H}$ associated with the optimization problem. One sufficient condition guaranteeing solutions satisfying the control problem are optimal is that $\mathbf{H}$ of the current value Hamiltonian is jointly concave in all the states and controls (Mangasarian, 1966). This is done if the determinant of $\mathbf{H}$ is strictly positive (the matrix is not negative semi-definite). While technical, Proposition 6 nicely states such condition. Remember that in CRRA setting, σ_1 and σ_2 are the coefficients of relative risk aversion. Then, σ_1 and σ_2 are the elasticities of marginal utility for consumption and health capital respectively. The sufficient optimality condition here is that the product of the relative risk aversion must be greater than the unit.

4.2 Equilibrium

The dynamics of the economy is driven at equilibrium by the following system

$$\begin{cases} \frac{\dot{\hat{C}}(z)}{\hat{C}(z)} = \frac{1}{\sigma_1}(\epsilon B\hat{k}_z^{\epsilon-1} - \delta - \rho) + \frac{1-\sigma_2}{\sigma_1}\frac{\dot{\hat{M}}(z)}{\hat{M}(z)} \\ \frac{\dot{\hat{m}}(z)}{\hat{m}(z)} = -\frac{1-\sigma_2}{1-\sigma_1}\frac{\pi\alpha\hat{m}^{\alpha-1}}{1-\alpha}\frac{\hat{C}(z)}{\hat{M}(z)} + \frac{1}{1-\alpha}(r+\delta_M) \\ \dot{\hat{M}}(z) = \pi(\hat{m}(z))^\alpha - \delta_M\hat{M}(z) \\ \dot{\hat{k}}_z = B\hat{k}_z^\epsilon - \hat{C} - \hat{m} - \delta\hat{k}_z \end{cases} \tag{40}$$

with $\hat{k}(0)$ and $\hat{M}(0)$ given, plus the transversality conditions. The steady-state values of $\hat{C}$, $\hat{m}$, $\hat{M}$, and $\hat{k}$ are obtained by equalising $\dot{\hat{C}}$, $\dot{\hat{m}}$, $\dot{\hat{M}}$, $\dot{\hat{k}}$ to zero. We therefore get:

$$\hat{C} = \frac{(\delta_M + r)(1-\sigma_1)B^{\frac{1}{1-\epsilon}}\left[\left(\frac{\delta+\rho}{\epsilon}\right)^{\frac{\epsilon}{\epsilon-1}} - \delta\left(\frac{\delta+\rho}{\epsilon}\right)^{\frac{1}{\epsilon-1}}\right]}{\alpha\delta_M(1-\sigma_2) + (1-\sigma_1)(\delta_M+r)} \tag{41}$$

$$\hat{m} = \frac{\alpha\delta_M(1-\sigma_2)B^{\frac{1}{1-\epsilon}}\left[\left(\frac{\delta+\rho}{\epsilon}\right)^{\frac{\epsilon}{\epsilon-1}} - \delta\left(\frac{\delta+\rho}{\epsilon}\right)^{\frac{1}{\epsilon-1}}\right]}{\alpha\delta_M(1-\sigma_2) + (1-\sigma_1)(\delta_M+r)} \tag{42}$$

$$\hat{M} = \frac{\pi\hat{m}^\alpha}{\delta_M} \tag{43}$$

$$\hat{k} = B^{\frac{1}{1-\epsilon}}\left(\frac{\delta+\rho}{\epsilon}\right)^{\frac{1}{\epsilon-1}} \tag{44}$$

Within the structure of the system, studying the stability properties analytically is simply unbearable. Instead, we resort to numerical simulations with a set of reasonable parameterizations and the necessary corroborating sensitivity tests. We always end up with four eigenvalues with two real negative parts and two real positive parts, suggesting that saddle path properties are fulfilled.[18]

4.3 Comparative statics

We now elaborate on how the health parameters (δ_M, π) affect the steady-state values, and notably the health investment variable ($\hat{m}$), and the consequences on consumption, capital stock and savings.

Proposition 7 *Given condition of Proposition (6), a high deterioration rate of health causes health expenditure to increase, and the latter is fully financed by a fall in consumption, and saving remains unaffected.*

Proof. See Appendix. □

Let us reiterate that with the benchmark model we get the same reaction only for σ_2 low enough, and more precisely when δ_M is high enough. On the contrary, in this case individuals react to health depreciation by spending more on health, independently of the size of σ_2, provided that the concavity requirements stated in proposition (6) are fulfilled. Without doubt, under the conditions of proposition (6) we have a positive effect of δ_M on $\hat{m}$. This positive effect can be

[18] To simplify presentation, we do not report simulation results. Results are available from the authors upon request.

explained by the fact that in the alternative framework, the marginal utility of consumption depends on health capital. Indeed, in order to enjoy consumption, one should be healthy, and a deterioration of health now negatively affects the marginal utility drawn from ordinary consumption. Therefore, one should invest more in health to maintain welfare optimal. This increase is paid by a fall in consumption, and since the individual's resources remain unchanged, if not improved, then there is no impact on gross investment in physical capital and savings remain constant.

Proposition 8 *Given Proposition (6), the effect of a decrease in the health productivity π on health investment, consumption and savings is neutral.*

Proof. The proof is intuitive. Indeed, a decrease in π has two effects as in the benchmark model: first, a fall in π decreases the marginal efficiency of health investment, therefore pushing this variable downward. But on the other hand, decreasing π also increases the marginal value of health capital relative to the marginal value of consumption, making it more desirable to acquire. However, contrarily to the benchmark model where for high values of σ_2 the negative effect dominates the positive one, in this alternative model, the second effect just offsets the first one. These two effects can be clearly identified in Equation (43) as well as the expression of the unit value of health capital $g = \frac{1}{\psi'(\hat{m})} = \frac{1}{\alpha\pi\hat{m}^{\alpha-1}}$. The rational of this neutral effect is that, here a decrease in health expenditure in response to a lower π (as it is the case in the benchmark model) is more harmful, since in turn we will have a negative impact on the marginal utility of consumption.

The conclusions emerging from the alternative model are different from those resulting from the benchmark model. The foremost and common result from the two models is that rising health expenditure does not always lead to a decrease in savings. However, while the benchmark model results completely rely on the health elasticity σ_2 as for the evolution of the health investment $\hat{m}$ in response to an health depreciation, in the alternative model, an increase in δ_M induces purely a growth in health expenditure. Moreover, we have a neutral effect when π decreases, without any elasticity constraint.

4.4 The case of productivity-decreasing health shock

As in the benchmark model, we take into account the health shock in the household's choices, via the channel of the productivity. We then assume that the level of technology $B(z)$ or the productivity of individuals depends on the health parameters. We suppose a function $B(\delta_M, \pi)$ which depends on δ_M and π. Then B is an increasing function in the health productivity π, and decreases with the health deterioration rate δ_M. Then, the following result can be highlighted

Proposition 9 *The effect of health deterioration rate on health investment is positive if and only if $\sigma_2 < 1 + \frac{(1-\sigma_1)\left[\delta_M B_{\delta_M}(\delta_M+\rho)+\rho(1-\epsilon)\right]}{\alpha\delta_M^2 B_{\delta_M}}$, and consumption and savings always decrease.*

Proof. See Appendix. □

In contrast to the case where the productivity B is independent of δ_M, here the optimality of increasing health expenditure in the face of health depreciation does depend on the values of σ_1 and σ_2. Clearly, the negative effect on productivity has an adverse effect on income, which in turn induces a decrease in ordinary consumption and health investment. Such effects can dominate those pointed out in the previous subsection where B is independent of health parameters.

This can be clearly deduced from the necessary and sufficient conditions in Proposition (9). In case of health deterioration with a tenus productivity effect, that is when B_{δ_M} is close to zero, the limit condition becomes $\sigma_2 < \infty$, that is $\hat{m}$ will increase whatever elasticities' values, provided that the Mangasarian condition is checked. In contrast, if health deterioration has strong adverse effects on productivity, say $B_{\delta_M} \to -\infty$, then the condition of Proposition (9) violates the Mangasarian restriction in Proposition (6). Indeed, the limit condition becomes $\sigma_2 < \frac{(1-\sigma_1)(\delta_M+\rho)}{\alpha\delta_M}$. Therefore $\sigma_2 < 1$ if and only if $\frac{(1-\sigma_1)(\delta_M+\rho)}{\alpha\delta_M} < 0$, which in turn implies that $\sigma_1 > 1$. Although the Mangasarian condition is a sufficient condition, the latter results mean that the range of elasticities' values yielding an increase in health expenditure will shrink as the adverse productivity effect becomes tougher. On the physical capital side, due to the negative effect on B, we have a fall in gross investment which causes savings to decrease. In this case, the role of public health expenditure becomes crucial.

Proposition 10 *In the presence of a decrease in π, health expenditure increases if and only if $\sigma_2 > 1$. Thus health expenditure should drop when π decreases in the alternative model, since $\sigma_2 < 1$ is imposed for marginal utility of M and C to be positive.*

Proof. See Appendix. $\square$

According to Proposition (6), Proposition (10) completely violates the concavity condition. Therefore, we cannot expect an increase in health expenditure when the health productivity π decreases. Indeed, the negative effect on the productivity undermines individuals resources, forcing them to undercut their health investment, consumption and savings. Here, the negative effect from the marginal efficiency of health dominates the positive one induced by the increase in the unit value of health. Indeed, as featured in Equation (38), marginal utilities with respect to M and C depend on the utility levels of C and M in this order (due to the non-separable preferences). Therefore the negative effect of decreasing efficiency in the production of health capital is magnified.

5 The equilibrium dynamics of the economy

This section is devoted to the analysis of the dynamical systems. This includes i) the study of the equilibrium dynamics with phase diagrams and ii) the characterization of possible bifurcations. Given the complexity of the problem, a first step was to reduce the dimension of the systems by moving from the original systems of four to three equations. This is done by expressing the variable in units of physical capital by setting: $c = \frac{C}{k}$, $\bar{m} = \frac{m}{k}$ and $\bar{M} = \frac{M}{k}$. As in previously, both separable and non-separable preferences are considered.

5.1 Separable preferences

5.1.1 Existence and stability of the system

In order to establish the new dynamical system, observe that

$$\frac{\dot{c}}{c} = \frac{\dot{C}}{C} - \frac{\dot{k}}{k} = \frac{r-\rho}{\sigma_1} - Bk^{\epsilon-1} + c + \bar{m} + \delta$$

However $r = B\epsilon k^{\epsilon-1} - \delta = B\epsilon k^{\epsilon-1} - \epsilon\delta + \epsilon\delta + \delta$. It follows that $B\epsilon k^{\epsilon-1} - \delta = \frac{r+\delta(1-\epsilon)}{\epsilon}$. Moreover, the variable $\bar{m}$ is implicit in the term

$$\begin{aligned}
\frac{\dot{\bar{m}}}{m} &= \frac{\dot{m}}{m} - \frac{\dot{k}}{k} \\
&= \frac{\delta_M + r}{1-\alpha} - b\bar{M}^{-\sigma_2}\pi\alpha\bar{m}^{\alpha-1}\left(\frac{r+\delta(1-\epsilon)}{\epsilon} - \frac{r-\rho}{\sigma_1} - \bar{m}\right)^{\sigma_1}\frac{1}{1-\alpha}\left(\frac{r+\delta}{\epsilon B}\right)^{\varpi} - \frac{r-\rho}{\sigma_1}
\end{aligned}$$

where $\varpi = \sigma_1 - \sigma_2 + \alpha - 1$ consequently, the variables follows the system:

$$\frac{\dot{c}}{c} = \frac{r-\rho}{\sigma_1} - \frac{r+\delta(1-\epsilon)}{\epsilon} + c + \bar{m} \tag{45}$$

$$\frac{\dot{\bar{m}}}{m} = \frac{\delta_M + r}{1-\alpha} - b\bar{M}^{-\sigma_2}\pi\alpha\bar{m}^{\alpha-1}c^{\sigma_1}\frac{1}{1-\alpha}\left(\frac{r+\delta}{\epsilon B}\right)^{\varpi} - \frac{r+\delta(1-\epsilon)}{\epsilon} + c + \bar{m} \tag{46}$$

$$\frac{\dot{\bar{M}}}{M} = \frac{\pi\bar{m}^{\alpha}}{\bar{M}}\left(\frac{r+\delta}{\epsilon B}\right)^{\alpha-1} - \delta_M - \frac{r+\delta(1-\epsilon)}{\epsilon} + c + \bar{m} \tag{47}$$

To solve the system given by Eq. (45-47), let us write $A_1 := \frac{1}{1-\alpha}\left(\frac{r+\delta}{\epsilon B}\right)^{\varpi}$, $A_2 := \left(\frac{r+\delta}{\epsilon B}\right)^{\alpha-1}$, $A_3 := \frac{\pi}{\delta_M + \frac{r-\rho}{\sigma_1}}A_2$, $A_4 := \frac{r+\delta(1-\epsilon)}{\epsilon} - \frac{r-\rho}{\sigma_1}$ and $A_5 := \frac{\delta_M + r}{1-\alpha} - \frac{r-\rho}{\sigma_1}$. After some calculations, the solution or steady state values of c, $\bar{M}$ and $\bar{m}$ are obtained as:

$$c^* = A_4 - \bar{m}^* \tag{48}$$

$$\bar{M}^* = A_3(\bar{m}^*)^{\alpha} \tag{49}$$

$$A_4(\bar{m}^*)^{-\sigma_2\alpha+\alpha-1} = \left(\frac{A_5}{A_1}\frac{A_3^{\sigma_2}}{b\pi\alpha}\right)^{\sigma_1^{-1}} + (\bar{m}^*)^{-\sigma_2\alpha+\alpha} \tag{50}$$

As can be seen from Eqs. (48-50) the steady for $(\bar{m}^*)$ is an implicit term. In what follows, we discuss the conditions of existence for the equilibrium and the stability of the system. The next proposition provides these conditions.

Proposition 11 *The dynamical system admits a unique saddle point solution, iff:*

i) $\rho - \delta_M\sigma_1 < r < \delta\sigma_1\frac{1-\epsilon}{1-\sigma_1}$

ii) $\frac{bc^{\sigma_1}}{\bar{M}^{\sigma_2}\bar{m}^2}\left(\frac{r+\delta}{\epsilon B}\right)^{\sigma_1-\sigma_2} > \frac{1}{\bar{M}^2}$

Proof. See Appendix. □

The condition i) in the proposition ensures the existence and uniqueness of the steady state whereas condition ii) guarantees the stability. Relying on the implicit behavior of $\bar{m}$ and parameters conditions, we prove in the Appendix that this leads to the the existence and uniqueness of the solution. Furthermore, the stability of the system is proved thanks to the analysis of the determinant and the trace of Jacobian matrix.

5.1.2 The dynamics of transition: phase diagrams

We are now interested in the changes in the equilibrium, when some health parameters are modified. We propose a geometrical representation by drawing the phase diagrams associated to the system (45-47). The diagrams are plotted on different planes while fixing each of the variables. Lemma 1 elaborates on the phase diagrams in Figures 1, 2 and 3.

Lemma 1 *We have:*

i) For $\bar{m}$ fixed, the curve $\dot{c} = 0$ is a horizontal line and the locus $\dot{\bar{M}} = 0$ is an increasing and concave function. Moreover we have $\lim_{\bar{M}\to\infty} c = b_2$ for b_2 given.

ii) For c fixed, the curve $\dot{\bar{M}} = 0$ monotonically increases and the locus $\dot{\bar{m}} = 0$ is a decreasing and convex function with $\lim_{\bar{m}\to 0} \bar{M} = 0$ and $\lim_{\bar{m}\to b^-} \bar{M} = \infty$.

iii) For $\bar{M}$ fixed, the curve $\dot{c} = 0$ is a decreasing straight line on $[0, d_0[$ and the locus $\dot{\bar{m}} = 0$ is an increasing function over the real support $[d_1, \bar{m}_0[$ and it decreases monotonically from $\bar{m}_0$.

Proof. See Appendix. □

Insert Figures 1, 2, 3

Fixing $\bar{m} = \bar{m}_0$ leads to the phase diagram onto the plane $(c, \bar{M})$ (see Figure 1). We have two curves. The first one ($\dot{c} = 0$) is independent of the stock of health. An increase of the rate of health depreciation (δ_M) implies a shift of the second curve ($\dot{\bar{M}}$) towards the left. This curve is increasing with both consumption and health. The shift induces a decrease of health from the steady state E_0 to a new one E_1, where consumption remains constant.

Fixing c gives the second diagram on the plane $(\bar{m}, \bar{M})$ (see Figure 2). The trade-off is between health stock and investment in health. The curve $\dot{\bar{M}} = 0$ is increasing but health grows more rapidly than the flow of investment. There is stability in the locus given by the right side of $\dot{\bar{M}} = 0$. When δ_M increases, the depreciation of health implies a lower level of the stock of health. The curve $\dot{\bar{m}} = 0$ shifts to the left. But the flow of investment increases, meaning that the final steady state is reached when the curve $\dot{\bar{M}} = 0$ moves forward. The final steady state is the point E_1 where the stock of health is lower than the one in the first equilibrium E_0.

Fixing $\bar{M}$ gives the third diagram in Figure (3) on the plane $(c, \bar{m})$. Relying on the equation ($\dot{c} = 0$) there is a linear relation between the flow of investment and consumption. The stable manifold is in the zones on the left hand side of E_0 above the curve and the right hand side of E_0 below the curve. Within these two zones, the trajectories converge to the steady state values. An increase of δ_M generates a higher level of investment flows. Moreover, a crowding effect appears and consumption jumps backward from the first steady state E_0 to the second one E_1.

5.2 Non-separable preferences

5.2.1 Existence and stability of the system

In that case and using the same variable definition above, the dynamical system is computed as:

$$\frac{\dot{c}}{c} = \frac{r - \delta\epsilon - \rho}{\sigma_1} - \frac{r + \delta(1-\epsilon)}{\epsilon} + \frac{1-\sigma_2}{\sigma_1}\pi\left(\frac{r+\delta}{\epsilon B}\right)^{\frac{\alpha-1}{\epsilon-1}} \bar{m}^{\alpha} + c + \bar{m} - \delta_M \tag{51}$$

$$\frac{\dot{\bar{m}}}{m} = -\frac{1-\sigma_2}{1-\sigma_1}\pi\frac{\alpha}{1-\alpha}\left(\frac{r+\delta}{\epsilon B}\right)^{\frac{\alpha-1}{\epsilon-1}} \frac{\bar{m}^{\alpha}}{\bar{M}} + c + \bar{m} + \frac{r+\delta_M}{1-\alpha} - \frac{r+\delta(1-\epsilon)}{\epsilon} \tag{52}$$

$$\frac{\dot{\bar{M}}}{\bar{M}} = \pi\left(\frac{r+\delta}{\epsilon B}\right)^{\frac{\alpha-1}{\epsilon-1}} \frac{\bar{m}^{\alpha}}{\bar{M}} + c + \bar{m} - \delta_M - \frac{r+\delta(1-\epsilon)}{\epsilon} \tag{53}$$

In order to simplify notations, let us denote $X_0 := \frac{r-\delta\epsilon-\rho}{\sigma_1} - \frac{r+\delta(1-\epsilon)}{\epsilon}$, $X_1 := \frac{1-\sigma_2}{\sigma_1}\pi\left(\frac{r+\delta}{\epsilon B}\right)^{\frac{\alpha-1}{\epsilon-1}}$, $X_2 := -\frac{\alpha\sigma_1}{(1-\sigma_1)(1-\alpha)}X_1$, $X_3 := \frac{r+\delta_M}{1-\alpha} - \frac{r+\delta(1-\epsilon)}{\epsilon}$, $X_4 := \pi\left(\frac{r+\delta}{\epsilon B}\right)^{\frac{\alpha-1}{\epsilon-1}}$, $X_5 := -\delta_M - \frac{r+\delta(1-\epsilon)}{\epsilon}$. The computation of the solution leads to the steady state values:

$$\bar{m}^* = X_1^{-\frac{1}{\alpha}}\left(\delta_M - X_0 + X_3 + X_2\frac{X_5 - X_3}{X_2 - X_4}\right)^{\frac{1}{\alpha}} \tag{54}$$

$$c^* = -\bar{m}^* - X_5 - X_4\frac{X_5 - X_3}{X_2 - X_4} \tag{55}$$

$$\bar{M}^* = (\bar{m}^*)^{\alpha}\frac{X_2 - X_4}{X_5 - X_3} \tag{56}$$

The next proposition provides existence conditions of the above mentioned equilibrium.

Proposition 12 *The following conditions must be verified for the existence of the equilibrium of the dynamical system:*

i) $1 < \sigma_1 < \frac{1-\alpha\sigma_2}{1-\alpha}$ *which implies* $\sigma_2 < 1$

ii) $\frac{1-\alpha\sigma_2}{1-\alpha} < \sigma_1 < 1$ *which implies* $\sigma_2 > 1$

Conditions i) and ii) implies the existence of a node and a saddle point respectively.

Proof. See Appendix. □

Thanks to the analysis of the Jacobian matrix associated to the system (51-53), it is demonstrated in the Appendix that if condition i) holds, the determinant of Jacobian matrix is always positive implying a node. If condition ii) holds, then the determinant is negative and there exists a saddle point. The stability of the node and the saddle point depends on the sign of the trace of the Jacobian matrix.

5.2.2 The dynamics of transition: phase diagrams

Here again, we turn to analysis of the changes in the equilibrium when some health parameters are modified. Lemma 2 documents on the phase diagrams in Figures 4, 5 and 6 which are associated to the system (51-53).

Lemma 2 *We have:*

i) For $\bar{m}$ *fixed, the curve* $\dot{c} = 0$ *is a horizontal line and the locus* $\dot{\bar{M}} = 0$ *is an increasing and concave function, with* $\lim_{\bar{M}\to\infty} c = a'$ *for* a' *given.*

ii) For c *fixed, the curves* $\dot{\bar{m}} = 0$ *and* $\dot{\bar{M}} = 0$ *are convex and monotonically increasing with* $\lim_{\bar{m}\to a} \bar{M} = \infty$ *and* $\lim_{\bar{m}\to a'} \bar{M} = \infty$ *for* a *and* a' *given.*

iii) For $\bar{M}$ *fixed, the curve* $\dot{c} = 0$ *is decreasing and the locus* $\dot{\bar{m}} = 0$ *is an increasing function over the support* $[d_1, m_1[$ *and it decreases monotonically from* m_1.

Proof. See Appendix. □

Insert Figures 4, 5, 6

We fix $\bar{m}$ and get the first diagram in Figure (4) onto the plane $(c, \bar{M})$. As for the additive case, the curve $\dot{c} = 0$ doesn't dependent on the stock of health $\bar{M}$. The consumption depends though only on the flow of investment and on the rate of health depreciation. The curve $\dot{\bar{M}} = 0$ is concave and increases with both variables. Starting from the steady state E_0, an increase of δ_M shifts upward the level of consumption, leading to the final equilibrium point E_1. The depreciation of health reduces the stock of health, but this reduction is compensated by the shift of consumption. The result is a net increase of health.

The second phase diagram in Figure (5) is plotted onto the plane $(\bar{m}, \bar{M})$ by fixing c. The two curves are increasing with the variables $\bar{m}$ and $\bar{M}$), but the stock of health grows faster than the flow of investment. An increasing rate of health depreciation leads to a high reduction of the stock of health which is not fully compensated by the investment. So the two curves shift down to the steady state E_1.

The third diagram in Figure (6) is obtained onto the plane $(c, \bar{m})$ by fixing $\bar{M}$. There is a linear relation between c and $\bar{m}$ that gives the monotonically decreasing curve. An increase of the depreciation rate shifts upward the consumption and the flow of investment in health on the curve $\dot{c} = 0$. We have the same with the curve $\dot{\bar{m}} = 0$. The final steady state is reached at E_1 where both variables increase.

At this stage of the dynamics of transition, it is important to draw attention to a nice point. Propositions 11 and 12 give conditions on parameters for the existence and the stability of the systems depending on the type of preferences. Nevertheless one can find a common support parameters to both types of preferences. This common support is illustrated geometrically in Figure 7. Indeed, from the condition i) in Proposition 11 we see that the interest rate is bounded below and above by terms depending on the parameter σ_1. As a result, the scale of the interest rate largely depends on σ_1. We draw the two bounds by varying σ_1 and find the area where our parameters lay. The first curve is $\underline{\mathcal{B}} := \rho - \delta_M \sigma_1$. It's a decreasing line and it represents the lower bound of the interest rate. The second curve is $\overline{\mathcal{B}} := \delta \sigma_1 \frac{1-\epsilon}{1-\sigma_1}$. It is represented by an hyperbola with $\lim_{\sigma_1 \to 1} \overline{\mathcal{B}} = \infty$. The minimal interest rate is $r_{\min} = \rho - \delta_M$ when $\sigma_1 = 1$. The minimal value of σ_1 is

$$\sigma_1^{\min} = \frac{\rho + \delta_M + \delta - \delta\epsilon - \sqrt{(\rho + \delta_M + \delta - \delta\epsilon)^2 - 4\rho\delta_M}}{2\delta_M} \tag{57}$$

It is obtained as the point where the two curves intersect which is the admissible root of second degree polynomial equation in σ_1:

$$\rho - \delta_M \sigma_1 = \delta \sigma_1 \frac{1-\epsilon}{1-\sigma_1} \Longleftrightarrow \rho - (\rho + \delta_M + \delta - \delta\epsilon)\sigma_1 + \delta_M \sigma_1^2 \tag{58}$$

Eq. (58) admits two roots $(\sigma_2', \sigma_2'') = \frac{\rho + \delta_M + \delta - \delta\epsilon \mp \sqrt{(\rho + \delta_M + \delta - \delta\epsilon)^2 - 4\rho\delta_M}}{2\delta_M}$ of which σ_2'' corresponds to negative values of interest rates, and σ_2' denotes the minimal positive interest rate. In Figure 7 the common support parameter σ_1 for both separable and non-separable preferences is represented by the shaded area. This area becomes larger if $\rho < \delta_M$.

Insert Figures 7

6 Bifurcation analysis: Effects of health depreciation on the steady states

Bifurcation occurs when a small smooth change made to the parameter values (the bifurcation parameters) of a system causes a sudden 'qualitative' change in its behaviour. In the previous

section, when studying the local stability, we observed that under certain conditions, saddle and node points emerged from the dynamical system. In this section, wanted to know which kind of preference (separable or non-separable) could favor the emergence of unstable motions. We firstly characterise analytically the bifurcations and then implement simulation experiments.

6.1 Characterization of bifurcations

The characterization of bifurcations is done by studying the eigenvalues of a dynamical system. In fact, we would like to understand the relations between the parameters and the eigenvalues. At the neighborhood of the steady state, we apply the following linear system:

$$c(t) = c^* + a_1 e^{\lambda_1 t}\nu_{11} + a_2 e^{\lambda_2 t}\nu_{21} + a_3 e^{\lambda_3 t}\nu_{31} \tag{59}$$

$$\bar{m}(t) = \bar{m}^* + a_1 e^{\lambda_1 t}\nu_{12} + a_2 e^{\lambda_2 t}\nu_{22} + a_3 e^{\lambda_3 t}\nu_{32} \tag{60}$$

$$\bar{M}(t) = \bar{M}^* + a_1 e^{\lambda_1 t}\nu_{13} + a_2 e^{\lambda_2 t}\nu_{23} + a_3 e^{\lambda_3 t}\nu_{33} \tag{61}$$

where λ_i's denote eigenvalues, the ν_{ij}'s are the coordinates of the eigenvectors associated to the different eigenvalues. The constants a_i are to be evaluated. By defining $\nu_{31} > 0$ as unstable eigenvector, one can set $a_3 = 0$. Moreover, we can normalise the first coordinate of the eigenvectors: $\nu_{11} = \nu_{21} = \nu_{31} = 1$. The eigenvalues of the matrices are given by the equation:

$$\det(J - AI) = F(\lambda) = -\lambda^3 + Tr(J)\lambda^2 + f\lambda + \det(J) = 0 \tag{62}$$

where I denotes the identity matrix, J the Jacobian matrix of a given system, A the matrix of coefficient a_i and f is a coefficient that depends on the elements of the matrix. By setting $Tr(J) = t$ and $\det(J) = d$, relation (62) becomes $F(\lambda) = -\lambda^3 + t\lambda^2 + f\lambda + d = 0$. After some analytical calculations, the general form of the eigenvalues λ_i are obtained as:

$$\lambda_1 = \frac{A}{6} - \frac{B}{A} + \frac{t}{3} \tag{63}$$

$$\lambda_{2,3} = -\frac{A}{12} + \frac{B}{2A} + \frac{t}{3} \pm i\frac{\sqrt{3}}{2}\left(\frac{A}{6} + \frac{B}{A}\right) \tag{64}$$

with

$$A = \left(36ft + 108d + t^3 + 12\sqrt{-12f^3 - 3f^2t^2 + 54ft + 81d^2 + 12dt^3}\right)^{1/3}$$

and

$$B = -2\left(f + \frac{1}{3}t^2\right)$$

For the case of separable preference (additive utility model), as the determinant d is negative, the equilibrium is characterised either by a negative eigenvalue, or three negative ones, or by a real eigenvalue and two complex ones. It is easy to see that real eigenvalues emerge if the following equality condition holds: $A^2 = -6B$. Unfortunately it has been impossible to bring out any analytical appraisal of these eigenvalues in the case of separable preferences. Indeed, as can be observed from Eqs. (63) and (64), these terms are terrific: bear in mind that terms A and B already include the determinant d and the trace t of a matrix that involved the Jacobian matrix J (see the expression of J in Appendix, Proof of Proposition 11). Therefore the full analytical characterization of the eigenvalues is simply unbearable. Instead, we resort to numerical simulations according to a set of parameter values. This is done in the next section.

For the the non-separable preferences, fortunately, the expressions of the eigenvalues can be analytically evaluated. Indeed, the expression of the associated Jacobian (see Appendix, Proof of Proposition 12) is still manageable. After some calculations, we obtain real eigenvalues:

$$\lambda_1 = \frac{\alpha(1-\sigma_2)X_4}{\bar{M}^2(1-\alpha)(1-\sigma_1)}X^\alpha \tag{65}$$

$$\lambda_{2,3} = \frac{1}{2}\Bigg[1+\frac{1}{\sigma_1}a\alpha X^{\alpha-1} \tag{66}$$

$$\mp \sqrt{5+2\left(1+\frac{1}{\sigma_1}a\alpha X^{\alpha-1}\right)\left(c-1+X_5+X+\frac{1}{\sigma_1}aX^\alpha+\frac{r-\delta\epsilon-\rho}{\sigma_1}\right)}\Bigg] \tag{67}$$

where $a=(1-\sigma_2)\pi\left(\frac{r+\delta}{\epsilon B}\right)^{\frac{\alpha-1}{\epsilon-1}}$ and

$$X=\left(\frac{a}{\sigma_1}\right)^{-\frac{1}{\alpha}}\left[\left(\delta_M+\frac{r+\delta_M}{1-\alpha}\right)\left(1+\frac{\alpha(1-\sigma_2)}{\alpha(1-\sigma_2)+(1-\alpha)(1-\sigma_1)}\right)-\frac{r-\delta\epsilon-\rho}{\sigma_1}\right]^{\frac{1}{\alpha}}$$

In what follows, we implements simulation experiments to learn more about the relations between the parameters and the eigenvalues.

6.2 Simulation experiments

This section aims mainly at understanding the relations between the parameters and the eigenvalues. As previously pointed out, this is analytically unbearable. Remmember that the dynamical systems are guided by the equations (45-47) for the additive utility, and by (51-53) for multiplicative utility functions. Studying the stability supposes the linearization of the systems around their steady state values, for given parameters as given by Eqs. (59-61). We have already shown the conditions of existence for theses values. The issues we are concerned with are the number and stability of the steady states, the bifurcation points and the tracking of the steady solutions. We run the simulations with the parameters: $b=1$, $\pi=1$, $\sigma_1=0.7$, $\sigma_2=1.4$, $\alpha=0.5$, $\delta=0.02$, $\epsilon=0.75$, $B=1$ $\theta:=\delta_M=0.025$, $\rho=0.01$, $r=0.1$.

We can notice that the common feature of the two cases of additive and multiplicative utility is that consumption per unit of capital c is linear in the first equations. It is also linearly related to the equilibrium value of the flow of health investment $\bar{m}$. We use this to simplify the computations. In the case of additive utility, when we fix the level of c, a bifurcation (see Figure 8) point appears at the level $\bar{m}=0.34182$ for a value of 0.106% of the rate of health depreciation. The plot is the projection of a 3-dimensional surface onto the plane $\bar{m}-\delta_M$. This bifurcation implies that the dynamical system qualitatively changes when δ_M is equal to that value given the other parameters. Consequently there exists an homomorphism in $\mathbb{R}^n$ that applies the trajectories among themselves, for $\delta_M<0.10$. The trajectories for other values of δ_M are also qualitatively equivalent.

Insert Figure 8

As it has been shown, the eigenvalues of the additive case can be complex numbers. But those of the multiplicative one are always real. We track the solutions by the 'continuation method'. We do an analysis in terms of level and have a qualitative graphic that shows how the steady state solutions evolve as we change the parameter δ_M. For this, we plot the level curves, for the two cases when δ_M varies within a low level (near zero), and a high level (near unity). The flow of investment in health capital m is in the horizontal axis and the stock of health capital M is in the vertical axis.

Insert Figure 9

It appears that when there is a fairly weak depreciation in health (see Figure 9), its stock is high in the additive case. Health grows more rapidly than investment, from the levels ($m = 0.2283, M = 0.7411$). Inversely the multiplicative case shows a lower health stock that increases more proportionately than investment flow from the point ($m = 0.3923, M = 0.06325$). The multiplicative case seems then more realistic.

When we set the depreciation rate to its maximum (see Figure 10), the multiplicative case shows very low levels of health capital combined with little investments. Alternatively, it appears in the additive case that more investment is done by individuals, but without significant improvement of health stock. The investment flow is relatively high in the case additive case but the gains in terms of health are not very important. The stock of health grows faster than investment from the point ($m = 0.5229, M = 0.5497$). Here again, the non-separable kind preference (multiplicative) is provide more realistic pictures.

Insert Figure 10

Now in Figure (11), we depict the evolution of the variables c, m and M, by setting δ_M to its low level. Consumption c is on the x-axis, the stock of health capital M is on the z-axis (the vertical one) and the flow of investment in health capital m is on the y-axis. In the case of additive utility, when the depreciation rate is low, the stock of health decreases even if consumption and health investment increase together. For fixed low level of investment m with ($m = 0.01$), the stock of health remains high but it tends to decrease when consumption grows. The decrease of the stock of health is accelerated by the investment. That's seems unrealistic. In the case of multiplicative utility, the stock increases with consumption and investment.

Insert Figure 11

Even when the depreciation rate is high (9.5%), the evolution of the variables remains nearly identical to the situation of zero depreciation (see Figure 12), for the additive case. But changes appear in the multiplicative case, as without investment, there is a positive relation between health and consumption.

Insert Figure 12

7 Concluding discussion

We summarise the key results that emerge from our study and compare with prior studies. Our findings can be gathered in three main strands.

i) *When productivity is independent of health parameters*

In the case of an acceleration of the health deterioration rate (δ_M), in the benchmark model, the reaction of health investment entirely relies on the elasticity of the marginal utility of health σ_2. Low values of σ_2 (or when δ_M high enough) are compatible with increasing health expenditure, while high values of σ_2 are likely to make the marginal welfare cost of such investment prohibitive. Indeed, the lower σ_2, the less marginal utility with respect to health capital will also drop. In the alternative model, individuals react to such shock by increasing their health expenditure, independently of the size of σ_2. This

behaviour can be interpreted by the fact that marginal utility of consumption now depends on health capital.

When we consider a loss of health productivity, we have two effects. Firstly, a decrease in health productivity (π) induces a loss in health investment efficiency, which causes health expenditure to drop. That's what we call the negative effect. Secondly, a weak value of π increases the marginal value of health capital relative to the marginal value of consumption goods, which in turn induces an incentive to invest in health. This is the positive effect. In the benchmark model, the positive effect dominates the negative one for σ_2 low enough, while in the alternative model, one of the two effects compensates the other, and the resulting impact is neutral.

However, in both models, regardless of the two kind of shocks, a relevant result is contrary to that of Cuddington and Hancok (1994). They predicted a decrease in savings while health expenditure increases. In our framework, irrespective of the increase in health investment, savings remain unchanged. And in case it increases, the latter is fully financed by a cut in consumption.

ii) *When productivity depends on health parameters*

In the benchmark model, a high rate of health deterioration induces an increase in health expenditure only for very low values of σ_2. In that case, if health deterioration has large negative effects on productivity, health investment does not increase. However, in the alternative model, findings depend on whether health shocks have tenuous or large effects on productivity. In the former case, health expenditure will increase whatever the elasticities' values. In the case of health shocks strongly affecting productivity, the range of elasticities' values yielding an increase in health expenditure will shrink as the adverse productivity effect becomes tougher.

With a loss of health productivity (a decrease in π), while in the benchmark model we obtain an increase in health expenditure for σ_2 low enough, this expenditure should drop in the alternative model (with multiplicative interaction between health and consumption). Indeed, in the latter model, marginal utilities with respect to health capital and consumption depend on the utility levels of consumption and health. Therefore, the negative effect of decreasing efficiency in the production of health capital is magnified.

Finally, when productivity depends on health parameters, an increase in health expenditure is found to be accompanied with a loss in both consumption and savings, due to the loss of productivity.

iii) *The dynamics*

The study of the dynamics sheds new light on the functioning of the models. The dynamics of transition allowed us to obtain the very nice result the geometrical illustration of the common parameters support. We have shown that while both types of preferences were not related *a priori*, their transitional dynamics can be connected. The density of the space defining the support of common parameters depends on the natural biological deterioration rate δ_M and the discount rate ρ. With regard to endogenous fluctuations, we observe that the model with separable preferences generates bifurcations with chaotic trajectories, while the model with non-separable preferences possesses a stable global dynamics.

As evident from above, the assumption of Cuddington and Hancok (1994) is certainly not fulfilled. A health impairment which stimulates increasing health expenditure and weakens the income base, not only affects savings, but it also compromises the consumption capacity, as well as the human and physical capital of the economy, and also undercuts the process of economic development. In such cases, the role of public health expenditure becomes crucial in increasing the average health level of individuals, which depends on the quality of the health sector, which in turn depends on the amount of public resources devoted to this sector. This finding can be linked in some sense to study of Hazan and Zoabi (2006) who incorporated health into a model extending the basic Ben-Porath mechanism. The authors considered health and education as integrated inputs in the production of human capital. They showed that increased health can not only increase the return on quality but also the return on quantity so that the transition from stagnation to growth becomes possible.

Our finding as regard the neutrality of health productivity π is also consistent with Hazan and Zoabi (2006). Indeed, Hazan and Zoabi (2006) considered an homothetic utility function and found that in a framework in which longevity is neutral, health can induce quantity-quality tradeoff. As a result, improvements in health are more likely to generate quantityquality tradeoff than gains in longevity. This result was also reinforced by Bleakley (2006) who found a natural experiment that bridges between health and longevity.

While the study aims at providing a richer relationship between health expenditure and outcomes than the existing literature, several challenges remain to be addressed. Some other worthwhile extensions would be i) the role for preventive health care along with having the consumer's wage and working time dependent on health, ii) the effect of health investments on expected lifetime, iii) the heterogeneity of populations (e.g. young/old, insured/uninsured). Yet, the model developed in this paper is complex enough to incorporate all the potential extensions. iv) Finally, so far, we have studied the model under the two polar assumptions of separable and non-separable preferences.

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Appendix

Proof of proposition 2
With Equation (21) can be rewritten in terms of output as:

$$f = \hat{m}^{\frac{\alpha}{\sigma_2}+1-\alpha} - \left(\frac{\delta_M}{\pi}\right)^{\frac{1}{\sigma_2}} \frac{b\pi\alpha(B\hat{k}^{\epsilon} - \hat{m} - \delta\hat{k})^{\sigma_1}}{\delta_M + r} = 0$$

Relying on the the implicit functions Theorem we obtain:

$$\begin{aligned}
\frac{\partial \hat{m}}{\partial \delta_M} &= -\frac{\frac{\partial f}{\partial \delta_M}}{\frac{\partial f}{\partial \hat{m}}} \\
&= \frac{\left(\frac{\delta_M}{\pi}\right)^{\frac{1}{\sigma_2}} b\pi\alpha(B\hat{k}^{\epsilon} - \hat{m} - \delta\hat{k})^{\sigma_1} \left(\frac{\delta_M+r}{\sigma_2\delta_M} - 1\right)}{(\delta_M + r)\left[(\frac{\alpha}{\sigma_2}+1-\alpha)m^{\frac{\alpha}{\sigma_2}-\alpha}(\delta_M + r) + \sigma_1\left(\frac{\delta_M}{\pi}\right)^{\frac{1}{\sigma_2}} b\pi\alpha(B\hat{k}^{\epsilon} - \hat{m} - \delta\hat{k})^{\sigma_1-1}\right]}
\end{aligned}$$

Now, observe that $\frac{\partial \hat{m}}{\partial \delta_M} > 0$ if $\frac{\delta_M+r}{\sigma_2\delta_M} - 1 > 0$, which means that $\sigma_2 > \frac{\delta_M}{\delta_M+\rho}$. Proceeding in the same way with respect to $\hat{C}$, we have:

$$\begin{aligned}
\frac{\partial \hat{C}}{\partial \delta_M} &= -\frac{\frac{\partial f}{\partial \delta_M}}{\frac{\partial f}{\partial \hat{C}}} \\
&= -\frac{\left(\frac{\delta_M}{\pi}\right)^{\frac{1}{\sigma_2}} b\pi\alpha\hat{C}^{\sigma_1} \left(\frac{\delta_M+r}{\sigma_2\delta_M} - 1\right)}{(\delta_M + r)\left[(\frac{\alpha}{\sigma_2}+1-\alpha)m^{\frac{\alpha}{\sigma_2}-\alpha}(\delta_M + r) + \sigma_1\left(\frac{\delta_M}{\pi}\right)^{\frac{1}{\sigma_2}} b\pi\alpha(B\hat{k}^{\epsilon} - \hat{m} - \delta\hat{k})^{\sigma_1-1}\right]}
\end{aligned}$$

In that case, if $\sigma_2 = \sigma_1 = 1$, and then:

$$\hat{m} = \frac{\delta_M b\alpha(B\hat{k}^{\epsilon} - \delta\hat{k})}{\delta_M(1 + b\alpha) + r}$$

It follows that:

$$\frac{\partial \hat{m}}{\partial \delta_M} = \frac{b\alpha(B\hat{k}^{\epsilon} - \delta\hat{k})\left[\delta_M(1 + b\alpha) + r\right] - \delta_M b\alpha(B\hat{k}^{\epsilon} - \delta\hat{k})(1 + b\alpha)}{\left[\delta_M(1 + b\alpha) + r\right]^2}$$

As a result, $\frac{\partial \hat{m}}{\partial \delta_M} > 0$ if and only if $r > 0$, which is always justified, and $\frac{\partial \hat{C}}{\partial \delta_M} = -\frac{\partial \hat{m}}{\partial \delta_M}$ $\square$

Proof of proposition 3
The derivative of $\hat{m}$ with respect to π is given by

$$\begin{aligned}
\frac{\partial \hat{m}}{\partial \pi} &= -\frac{\frac{\partial f}{\partial \pi}}{\frac{\partial f}{\partial \hat{m}}} \\
&= \frac{\left(\frac{\delta_M}{\pi}\right)^{\frac{1}{\sigma_2}} b\alpha(B\hat{k}^{\epsilon} - \hat{m} - \delta\hat{k})^{\sigma_1} \left(1 - \frac{1}{\sigma_2}\right)}{(\frac{\alpha}{\sigma_2}+1-\alpha)m^{\frac{\alpha}{\sigma_2}-\alpha}(\delta_M + r) + \sigma_1\left(\frac{\delta_M}{\pi}\right)^{\frac{1}{\sigma_2}} b\pi\alpha(B\hat{k}^{\epsilon} - \hat{m} - \delta\hat{k})^{\sigma_1-1}}
\end{aligned}$$

Then $\frac{\partial \hat{m}}{\partial \pi} < 0$ if and only if $\sigma_2 < 1$, which ends the first part of the proof. Similarly, we have:

$$\begin{aligned}
\frac{\partial \hat{C}}{\partial \pi} &= -\frac{\frac{\partial f}{\partial \pi}}{\frac{\partial f}{\partial \hat{C}}} \\
&= -\frac{\left(\frac{\delta_M}{\pi}\right)^{\frac{1}{\sigma_2}} b\alpha(B\hat{k}^{\epsilon} - \hat{m} - \delta\hat{k})^{\sigma_1} \left(1 - \frac{1}{\sigma_2}\right)}{(\frac{\alpha}{\sigma_2}+1-\alpha)m^{\frac{\alpha}{\sigma_2}-\alpha}(\delta_M + r) + \sigma_1\left(\frac{\delta_M}{\pi}\right)^{\frac{1}{\sigma_2}} b\pi\alpha(B\hat{k}^{\epsilon} - \hat{m} - \delta\hat{k})^{\sigma_1-1}}
\end{aligned}$$

Therefore, $\frac{\partial \hat{C}}{\partial \pi} > 0$ if and only if $\sigma_2 < 1$. If $\sigma_2 = \sigma_2 = 1$, $\frac{\partial \hat{m}}{\partial \pi} = \frac{\partial \hat{C}}{\partial \pi} = 0$ $\square$

Proof of proposition 4

Recall that we are dealing with the case of productivity B depending on health productivity and π and the health depreciation rate δ_M. Thus, we have

$$\begin{aligned}
\frac{\partial \hat{m}}{\partial \delta_M} &= -\frac{\frac{\partial f}{\partial \delta_M}}{\frac{\partial f}{\partial \hat{m}}} \\
&= \frac{\left(\frac{\delta_M}{\pi}\right)^{\frac{1}{\sigma_2}} b\pi\alpha(B\hat{k}^{\epsilon} - \hat{m} - \delta\hat{k})^{\sigma_1}\left[\frac{\delta_M + r}{\sigma_2\delta_M} - 1 + B_{\delta_M}\sigma_1\hat{k}^{\epsilon}(\delta_M + r)(B\hat{k}^{\epsilon} - \hat{m} - \delta\hat{k})^{-1}\right]}{(\delta_M + r)\left[(\frac{\alpha}{\sigma_2} + 1 - \alpha)m^{\frac{\alpha}{\sigma_2} - \alpha}(\delta_M + r) + \sigma_1\left(\frac{\delta_M}{\pi}\right)^{\frac{1}{\sigma_2}} b\pi\alpha(B\hat{k}^{\epsilon} - \hat{m} - \delta\hat{k})^{\sigma_1 - 1}\right]}
\end{aligned}$$

It then follows that $\frac{\partial \hat{m}}{\partial \delta_M} > 0$ if and only if $\frac{\delta_M + r}{\sigma_2\delta_M} - 1 + B_{\delta_M}\sigma_1\hat{k}^{\epsilon}(\delta_M + r)(B\hat{k}^{\epsilon} - \hat{m} - \delta\hat{k})^{-1} > 0$, that is $\sigma_2 < \frac{1}{1 - B_{\delta_M}\sigma_1\hat{k}^{\epsilon}(\delta_M + r)(B\hat{k}^{\epsilon} - \hat{m} - \delta\hat{k})^{-1}} + \frac{\rho}{\delta_M - \delta_M B_{\delta_M}\sigma_1\hat{k}^{\epsilon}(\delta_M + r)(B\hat{k}^{\epsilon} - \hat{m} - \delta\hat{k})^{-1}}$ $\square$ Also,

$$\begin{aligned}
\frac{\partial \hat{C}}{\partial \delta_M} &= -\frac{\frac{\partial f}{\partial \delta_M}}{\frac{\partial f}{\partial \hat{C}}} \\
&= -\frac{\left(\frac{\delta_M}{\pi}\right)^{\frac{1}{\sigma_2}} b\pi\alpha(B\hat{k}^{\epsilon} - \hat{m} - \delta\hat{k})^{\sigma_1}\left[\frac{\delta_M + r}{\sigma_2\delta_M} - 1 + B_{\delta_M}\sigma_1\hat{k}^{\epsilon}(\delta_M + r)(B\hat{k}^{\epsilon} - \hat{m} - \delta\hat{k})^{-1}\right]}{(\delta_M + r)\left[(\frac{\alpha}{\sigma_2} + 1 - \alpha)m^{\frac{\alpha}{\sigma_2} - \alpha}(\delta_M + r) + \sigma_1\left(\frac{\delta_M}{\pi}\right)^{\frac{1}{\sigma_2}} b\pi\alpha(B\hat{k}^{\epsilon} - \hat{m} - \delta\hat{k})^{\sigma_1 - 1}\right]} \quad \square
\end{aligned}$$

Proof of proposition 5

We have:

$$\begin{aligned}
\frac{\partial \hat{m}}{\partial \pi} &= -\frac{\frac{\partial f}{\partial \pi}}{\frac{\partial f}{\partial \hat{m}}} \\
&= \frac{\left(\frac{\delta_M}{\pi}\right)^{\frac{1}{\sigma_2}} b\alpha(B\hat{k}^{\epsilon} - \hat{m} - \delta\hat{k})^{\sigma_1}\left[1 - \frac{1}{\sigma_2} + B_{\pi}\sigma_1\hat{k}^{\epsilon}\pi(B\hat{k}^{\epsilon} - \hat{m} - \delta\hat{k})^{-1}\right]}{(\frac{\alpha}{\sigma_2} + 1 - \alpha)m^{\frac{\alpha}{\sigma_2} - \alpha}(\delta_M + r) + \sigma_1\left(\frac{\delta_M}{\pi}\right)^{\frac{1}{\sigma_2}} b\pi\alpha(B\hat{k}^{\epsilon} - \hat{m} - \delta\hat{k})^{\sigma_1 - 1}}
\end{aligned}$$

Then we can see easily that $\frac{\partial \hat{m}}{\partial \pi} < 0$ if and only if $1 - \frac{1}{\sigma_2} + B_{\pi}\sigma_1\hat{k}^{\epsilon}\pi(B\hat{k}^{\epsilon} - \hat{m} - \delta\hat{k})^{-1} < 0$, that is $\sigma_2 < \frac{1}{1 + B_{\pi}\sigma_1\hat{k}^{\epsilon}\pi(B\hat{k}^{\epsilon} - \hat{m} - \delta\hat{k})^{-1}}$. Also,

$$\begin{aligned}
\frac{\partial \hat{C}}{\partial \pi} &= -\frac{\frac{\partial f}{\partial \pi}}{\frac{\partial f}{\partial \hat{C}}} \\
&= -\frac{\left(\frac{\delta_M}{\pi}\right)^{\frac{1}{\sigma_2}} b\alpha(B\hat{k}^{\epsilon} - \hat{m} - \delta\hat{k})^{\sigma_1}\left[1 - \frac{1}{\sigma_2} + B_{\pi}\sigma_1\hat{k}^{\epsilon}\pi(B\hat{k}^{\epsilon} - \hat{m} - \delta\hat{k})^{-1}\right]}{(\frac{\alpha}{\sigma_2} + 1 - \alpha)m^{\frac{\alpha}{\sigma_2} - \alpha}(\delta_M + r) + \sigma_1\left(\frac{\delta_M}{\pi}\right)^{\frac{1}{\sigma_2}} b\pi\alpha(B\hat{k}^{\epsilon} - \hat{m} - \delta\hat{k})^{\sigma_1 - 1}} \quad \square
\end{aligned}$$

Proof of proposition 6

The Mangasarian sufficient condition stated is obtained by establishing the concavity of the objective function in (C, M). The concavity of the state functions and the positivity of λ_A and λ_M are straightforward. The Hamiltonian is given by

$$\begin{aligned}
\mathcal{H} &= U(C(z))\varphi(M(z))e^{-\rho z} \\
&+ \lambda_A e^{-\rho z}\left[r(z)A(z) + w(z) - C(z) - m(z)\right] \\
&+ \lambda_M e^{-\rho z}\left[\psi(m(z)) - \delta_M M(z)\right]
\end{aligned}$$

Let $\mathbf{H} = [\mathcal{H}(C, M, m, A, r, w)]$ be the Hessian matrix of $\mathcal{H}$ and let $|\mathbf{H}|$ denotes the determinant of $\mathbf{H}$. After deriving the second derivatives of $\mathcal{H}$, it is straightforward to show that the strict positivity of $|\mathbf{H}|$ requires

$$|\mathbf{H}| > 0 \Longleftrightarrow C^{-2\sigma_1} M^{-2\sigma_2} \left[\frac{\sigma_1\sigma_2}{(1-\sigma_1)(1-\sigma_2)} - 1\right] > 0$$

That is either $\sigma_1 < 1$ and $\sigma_2 < 1$ or $\sigma_1 > 1$ and $\sigma_2 > 1$ □

Proof of proposition 7

The proof is based on the sign of the derivatives $\frac{\partial \hat{m}}{\partial \delta_M}$ and $\frac{\partial \hat{C}}{\partial \delta_M}$. We have:

$$\frac{\partial \hat{m}}{\partial \delta_M} = \frac{r\alpha B^{\frac{1}{1-\epsilon}} \left[\left(\frac{\rho+\delta}{\epsilon}\right)^{\frac{\epsilon}{\epsilon-1}} - \delta \left(\frac{\rho+\delta}{\epsilon}\right)^{\frac{1}{\epsilon-1}}\right] (1-\sigma_2)(1-\sigma_1)}{\left[\alpha\delta_M (1-\sigma_2) + (1-\sigma_1)(\delta_M + r)\right]^2} > 0$$

and,

$$\frac{\partial \hat{C}}{\partial \delta_M} = -\frac{r\alpha B^{\frac{1}{1-\epsilon}} \left[\left(\frac{\rho+\delta}{\epsilon}\right)^{\frac{\epsilon}{\epsilon-1}} - \delta \left(\frac{\rho+\delta}{\epsilon}\right)^{\frac{1}{\epsilon-1}}\right] (1-\sigma_2)(1-\sigma_1)}{\left[\alpha\delta_M (1-\sigma_2) + (1-\sigma_1)(\delta_M + r)\right]^2} < 0 \quad \square$$

Proof of proposition 9

We have:

$$\begin{aligned}
\frac{\partial \hat{m}}{\partial \delta_M} &= \frac{\alpha\delta_M (1-\sigma_2) B^{\frac{\epsilon}{1-\epsilon}} \left[\left(\frac{\rho+\delta}{\epsilon}\right)^{\frac{\epsilon}{\epsilon-1}} - \delta \left(\frac{\rho+\delta}{\epsilon}\right)^{\frac{1}{\epsilon-1}}\right] B_{\delta_M} \left[\alpha\delta_M (1-\sigma_2) + (1-\sigma_1)(\delta_M + r)\right]}{(1-\epsilon)\left[\alpha\delta_M (1-\sigma_2) + (1-\sigma_1)(\delta_M + r)\right]^2} \\
&+ \frac{r\alpha B^{\frac{1}{1-\epsilon}} \left[\left(\frac{\rho+\delta}{\epsilon}\right)^{\frac{\epsilon}{\epsilon-1}} - \delta \left(\frac{\rho+\delta}{\epsilon}\right)^{\frac{1}{\epsilon-1}}\right] (1-\sigma_2)(1-\sigma_1)}{\left[\alpha\delta_M (1-\sigma_2) + (1-\sigma_1)(\delta_M + r)\right]^2}
\end{aligned}$$

and,

$$\begin{aligned}
\frac{\partial \hat{C}}{\partial \delta_M} &= \frac{(\delta_M + r)_M (1-\sigma_1) B^{\frac{\epsilon}{1-\epsilon}} \left[\left(\frac{\rho+\delta}{\epsilon}\right)^{\frac{\epsilon}{\epsilon-1}} - \delta \left(\frac{\rho+\delta}{\epsilon}\right)^{\frac{1}{\epsilon-1}}\right] B_{\delta_M} \left[\alpha\delta_M (1-\sigma_2) + (1-\sigma_1)(\delta_M + r)\right]}{(1-\epsilon)\left[\alpha\delta_M (1-\sigma_2) + (1-\sigma_1)(\delta_M + r)\right]^2} \\
&- \frac{r\alpha B^{\frac{1}{1-\epsilon}} \left[\left(\frac{\rho+\delta}{\epsilon}\right)^{\frac{\epsilon}{\epsilon-1}} - \delta \left(\frac{\rho+\delta}{\epsilon}\right)^{\frac{1}{\epsilon-1}}\right] (1-\sigma_2)(1-\sigma_1)}{\left[\alpha\delta_M (1-\sigma_2) + (1-\sigma_1)(\delta_M + r)\right]^2} < 0 \quad \square
\end{aligned}$$

Proof of proposition 10

The conditions ate checked for $\frac{\partial \hat{m}}{\partial \pi}$, and $\frac{\partial \hat{C}}{\partial \pi}$ as:

$$\frac{\partial \hat{m}}{\partial \pi} = \frac{\alpha\delta_M (1-\sigma_2) B^{\frac{\epsilon}{1-\epsilon}} \left[\left(\frac{\rho+\delta}{\epsilon}\right)^{\frac{\epsilon}{\epsilon-1}} - \delta \left(\frac{\rho+\delta}{\epsilon}\right)^{\frac{1}{\epsilon-1}}\right] B_\pi}{(1-\epsilon)\left[\alpha\delta_M (1-\sigma_2) + (1-\sigma_1)(\delta_M + r)\right]^2}$$

So $\frac{\partial \hat{m}}{\partial \pi} < 0$ if and only if $1 - \sigma_2 < 0$, then $\sigma_2 > 1$. Similarly,

$$\frac{\partial \hat{C}}{\partial \pi} = -\frac{(\delta_M + r)(1-\sigma_1) B^{\frac{\epsilon}{1-\epsilon}} \left[\left(\frac{\rho+\delta}{\epsilon}\right)^{\frac{\epsilon}{\epsilon-1}} - \delta \left(\frac{\rho+\delta}{\epsilon}\right)^{\frac{1}{\epsilon-1}}\right] B_\pi}{(1-\epsilon)\left[\alpha\delta_M (1-\sigma_2) + (1-\sigma_1)(\delta_M + r)\right]^2} \quad \square$$

Proof of proposition 11

The condition i) in the proposition ensures the existence and uniqueness of equilibrium, while condition ii) guarantees the stability. Indeed, the right hand side of Eq. (50) is an increasing function of $\bar{m}$ while the left hand side strictly decreases as long as the parameters σ_1, σ_2 and α are less than unity. As a result, A_4 must be positive. Also, from the fact that A_3 in Eq. (49) must be positive, the existence and uniqueness of the equilibrium follow. To prove the stability of the system (45-47), we evaluate its Jacobian matrix:

$$J = \begin{pmatrix} 1 & 1 & 0 \\ -b\sigma_1 \bar{M}^{-\sigma_2}\pi\alpha\bar{m}^{\alpha-1}A_1 c^{\sigma_1-1}+1 & -b\bar{M}^{-\sigma_2}\pi\alpha(\alpha-1)\bar{m}^{\alpha-2}A_1c^{\sigma_1}+1 & b\sigma_2\bar{M}^{-\sigma_2-1}\pi\alpha\bar{m}^{\alpha-1}A_1c^{\sigma_1} \\ 1 & \bar{M}^{-1}\pi\alpha\bar{m}^{\alpha-1}A_2+1 & -\bar{M}^{-2}\pi\alpha\bar{m}^{\alpha}A_2 \end{pmatrix}$$

The determinant of J is computed as:

$$\begin{aligned} \det(J) &= -b(\pi\alpha)^2 A_1 A_2 \bar{M}^{-\sigma_2-2}\bar{m}^{-1}c^{\sigma_1}\left(\frac{1-\alpha}{\bar{m}}+\frac{\sigma_1}{c}\right) \\ &- \left(\bar{M}^{-1}\pi\alpha\bar{m}^{\alpha-1}A_2\right)\left(b\sigma_2\bar{M}^{-\sigma_2-1}\pi\alpha\bar{m}^{\alpha-1}A_1c^{\sigma_1}\right) \end{aligned}$$

which is negative implying the existence of a saddle point. The stability depends on the sign of the trace of the matrix. The trace is computed as

$$\begin{aligned} Tr(J) &= 2 + b\bar{M}^{-\sigma_2}\pi\alpha(1-\alpha)\bar{m}^{\alpha-2}A_1c^{\sigma_1} - \bar{M}^{-2}\pi\alpha\bar{m}^{\alpha}A_2 \\ &= 2 + \pi\alpha\bar{m}^{\alpha}\left(\frac{r+\delta}{\epsilon B}\right)^{\alpha-1}\left[\frac{bc^{\sigma_1}}{\bar{M}^{\sigma_2}\bar{m}^2}\left(\frac{r+\delta}{\epsilon B}\right)^{\sigma_1-\sigma_2} - \frac{1}{\bar{M}^2}\right] \end{aligned}$$

which is positive if $\frac{bc^{\sigma_1}}{\bar{M}^{\sigma_2}\bar{m}^2}\left(\frac{r+\delta}{\epsilon B}\right)^{\sigma_1-\sigma_2} > \frac{1}{\bar{M}^2}$. $\square$

Proof of proposition 12

The Jacobian matrix of the system (51-53) is:

$$J = \begin{pmatrix} 1 & \alpha X_1 + 1 & 0 \\ 1 & \bar{M}^{-1}\bar{m}^{\alpha-1}X_2+1 & -\bar{M}^{-2}\bar{m}^{\alpha}X_2 \\ 1 & \bar{M}^{-1}\bar{m}^{\alpha-1}X_4+1 & -\bar{M}^{-2}\bar{m}^{\alpha}X_4 \end{pmatrix}$$

The determinant of J is computed as:

$$\det(J) = -\frac{\alpha}{\sigma_1}(1-\sigma_2)\bar{M}^{-2}\bar{m}^{\alpha}X_4^2\left(\frac{1-\sigma_2}{1-\sigma_1}\frac{\alpha}{1-\alpha}+1\right)$$

One easily checks that $\det(J)$ is positive condition if condition i) Proposition 12 holds and $\det(J)$ is negative if condition ii) holds. The trace is evaluated as

$$Tr(J) = 2 - (1-\sigma_2)\bar{M}^{-1}\bar{m}^{\alpha-1}X_4\left(\frac{1-\sigma_2}{1-\sigma_1}\frac{\alpha^2}{1-\alpha}\bar{M}^{-1}\bar{m}+1\right)$$

We see that if condition i) in the proposition holds, the trace is always positive if $\bar{m} < \frac{\sigma_1-1}{1-\sigma_2}\frac{1-\alpha}{\alpha^2}\bar{M}$. However, if condition ii) holds, the trace is positive if $\bar{m} > \frac{1-\sigma_1}{\sigma_2-1}\frac{1-\alpha}{\alpha^2}\bar{M}$ $\square$

Proof of Lemma 1

i) Figure (1): Relation (45) becomes constant and the (first) curve for $\dot{c} = 0$ is a straight line $b_1 = -\frac{r-\rho}{\sigma_1} + \frac{r+\delta(1-\epsilon)}{\epsilon} - \bar{m}$ defined on $\mathbb{R}^+$. Relation (47) gives the second curve $c = -\frac{a}{\bar{M}} + b_2$ with $a = \pi\bar{m}^{\alpha}\left(\frac{r+\delta}{\epsilon B}\right)^{\alpha-1}$ and $b_2 = \delta_M + \frac{r+\delta(1-\epsilon)}{\epsilon} - \bar{m}$. The support of c is $\mathbb{R}^*_+$.

The derivatives of c are such that $\frac{dc}{d\bar{M}} = \frac{a}{\bar{M}^2}$ and $\frac{d^2c}{d\bar{M}^2} = -\frac{2a}{\bar{M}^{-3}}$ meaning that the function c is increasing and concave. Moreover we have $\lim_{\bar{M}\to\infty} c = b_2$. Finally the values a_0 and a_1 in Figure (1) are computed as $a_0 = \frac{a}{b_2}$ and $a_1 = \frac{a}{\frac{r-\rho}{\sigma_1}+\delta_M}$. ii) Figure (2): The (first) curve for $\dot{\bar{m}} = 0$ is $\bar{M} = a'\left(\frac{\bar{m}^{\alpha-1}}{a+\bar{m}}\right)^{\frac{1}{\sigma_2}}$ is defined over the support $\mathbb{R}_+^*$ as $\bar{m} > 0$ with $a' = \frac{\delta_M+r}{1-\alpha} - \frac{r+\delta(1-\epsilon)}{\epsilon} + c$ and $a = b\pi\alpha c^{\sigma_1}\frac{1}{1-\alpha}\left(\frac{r+\delta}{\epsilon B}\right)^{\sigma_1-\sigma_2+\alpha-1}$. We easily check that $\frac{d\bar{M}}{d\bar{m}} = -\frac{1}{\sigma_2}\left[\bar{m}(2-\alpha)+a(1-\alpha)\right]\bar{m}^{\alpha}\left(\frac{\bar{m}^{\alpha-1}}{a+\bar{m}}\right)^{1+\frac{1}{\sigma_2}} < 0$ and

$$\begin{aligned}\frac{d^2\bar{M}}{d\bar{m}^2} &= -\frac{1}{\sigma_2}\Big[\bar{m}^2(2-\alpha)(2-\alpha+\sigma_2) - 2a\bar{m}(1-\alpha)(2-\alpha+\sigma_2)\\ &+ a^2(1-\alpha)(2-\alpha+\sigma_2)\Big]\frac{1}{\bar{m}^2(a+\bar{m})^2\sigma_2}\left(\frac{\bar{m}^{\alpha-1}}{a+\bar{m}}\right)^{\frac{1}{\sigma_2}} > 0\end{aligned}$$

meaning that the curve increases and is convex. The (second) function for $\dot{\bar{M}} = 0$ is $\bar{M} = a\frac{\bar{m}^{\alpha}}{b-\bar{m}}$ with $a = \pi\left(\frac{r+\delta}{\epsilon B}\right)^{\alpha-1}$ and $b = \frac{r+\delta(1-\epsilon)}{\epsilon} - c$. It is defined over the support $\mathbb{R}^+\backslash\{b\}$. We have $\frac{d\bar{M}}{d\bar{m}} = a\frac{\bar{m}^{\alpha-1}}{(b-\bar{m})^2}(\bar{m}+b\alpha-\bar{m}\alpha)$. $\frac{d\bar{M}}{d\bar{m}} > 0$ as $\alpha < 1$. $\frac{d^2\bar{M}}{d\bar{m}^2} = a\frac{\bar{m}^{\alpha-2}}{(b-\bar{m})^3}[\bar{m}^2(2-3\alpha+\alpha^2) - b^2\alpha(1-\alpha) + 2bm(2-\alpha)] > 0$. Furthermore, $\lim_{\bar{m}\to 0}\bar{M} = 0$ and $\lim_{\bar{m}\to b^-}\bar{M} = \infty$. As $\bar{M}$ is a positive function, we don't take into account the values for b^+. iii) Figure (3): The (first) curve for $\dot{c} = 0$ is a straight line $c = -\frac{r-\rho}{\sigma_1} + \frac{r+\delta(1-\epsilon)}{\epsilon} - \bar{m}$ which defined on $[0, d_0[$ with $d_0 = -\frac{r-\rho}{\sigma_1} + \frac{r+\delta(1-\epsilon)}{\epsilon}$. The (second) curve obtained for $\dot{\bar{m}} = 0$ and defined on $\mathbb{R}^+$ and is $c = a''\left(\frac{a-a'\bar{m}^{\alpha}}{\bar{m}^{\alpha-1}}\right)^{\frac{1}{\sigma_2}}$, with $a = \frac{\delta_M+r}{1-\alpha} + \delta_M$, $a' = \frac{\pi A_2}{\bar{M}}$, $a'' = \left(b\bar{M}^{\sigma_2}\pi\alpha A_1\right)^{-\frac{1}{\sigma_1}}$ and A_2 as defined in the text. We have $\frac{dc}{d\bar{m}} = a''\frac{(\bar{m}(a\bar{m}^{\alpha}-b))^{\frac{1}{\sigma_2}}}{\bar{m}\sigma_1(a'\bar{m}^{\alpha}-a)}(a'\bar{m}^{\alpha}-a+a\alpha) = 0 \Leftrightarrow \bar{m}_0 = \left(\frac{a(1-\alpha)}{a'}\right)^{\frac{1}{\alpha}}$. The function increases from d_1 to $\bar{m}_0$ and decreases decreases monotonically after. Moreover,

$$\begin{aligned}\frac{d^2c}{d\bar{m}^2} &= a''\frac{(\bar{m}(a\bar{m}^{\alpha}-b))^{\frac{1}{\sigma_2}}}{\bar{m}^2\alpha^2(a-a'\bar{m}^{\alpha})}[-a'^2\bar{m}^{2\alpha}(1-\alpha)\\ &- a^2(1-\alpha)(\alpha-\sigma_1-1) - ab\bar{m}^{\alpha}(1-\alpha)(2\sigma_1+\alpha\sigma_1-2)] < 0\end{aligned}$$

The other quantities in Figure (3) are $d_1 = \left(\frac{1}{\pi A_1}a\right)^{\frac{1}{\alpha}}$ and $C_0 = d_0 - \bar{m}_0$.

Proof of Lemma 2

i) Figure (4): For $\dot{\bar{m}} = 0$, the (first) curve is $\bar{M} = -X_2\frac{\bar{m}^{\alpha}}{a+\bar{m}}$ with $a = X_3 + c$. It is defined over the set $\mathbb{R}^+\backslash\{-a\}$. We have $\frac{d\bar{M}}{d\bar{m}} = X_2\frac{\bar{m}^{\alpha-1}}{(a+\bar{m})^2}(\bar{m}(1-\alpha)-a\alpha)$ which is positive iff $\bar{m} > \frac{a\alpha}{1-\alpha}$. Also, $\frac{d^2\bar{M}}{d\bar{m}^2} = -X_2\frac{\bar{m}^{\alpha-2}}{(a+\bar{m})^3}(2a\bar{m}(2-\alpha)+a^2\alpha(1-\alpha)-\bar{m}^2(2-3\alpha+\alpha^2)) > 0$. As a result $\bar{M}$ is convex and $\lim_{\bar{m}\to 0}\bar{M} = 0$ and $\lim_{\bar{m}\to -a}\bar{M} = \infty$. For $\dot{\bar{M}} = 0$ the (second) curve is $\bar{M} = -X_4\frac{\bar{m}^{\alpha}}{a'+\bar{m}}$ with $a' = X_5+c$. Its support is $\mathbb{R}^+\backslash\{-a'\}$. We have $\frac{d\bar{M}}{d\bar{m}} = X_4\frac{\bar{m}^{\alpha-1}}{(a'+\bar{m})^2}(\bar{m}(1-\alpha)-a'\alpha)$ which is positive iff $\bar{m} > \frac{a'\alpha}{1-\alpha}$. Furthermore, $\frac{d^2M}{d\bar{m}^2} = -X_4\frac{\bar{m}^{\alpha-2}}{(a'+\bar{m})^3}(2a'\bar{m}(2-\alpha)+a'^2\alpha(1-\alpha)-\bar{m}^2(2-3\alpha+\alpha^2)) > 0$. It follows that $\bar{M}$ is convex with $\lim_{\bar{m}\to 0}\bar{M} = 0$ and $\lim_{\bar{m}\to -a'}\bar{M} = \infty$. The quantity $\bar{m}_2$ in Figure (4) is obtained as follows. We have $-X_2\frac{\bar{m}^{\alpha}}{a+\bar{m}} = -X_4\frac{\bar{m}^{\alpha}}{a'+\bar{m}} \Leftrightarrow X_2a' - aX_4 = \bar{m}(X_4-X_2)$ implying that $\bar{m}_2 = \frac{X_2a'-X_4a}{X_4-X_2}$. ii) Figure (5): From relation (51), the (first) curve for $\dot{c} = 0$ is a straight line on $\mathbb{R}^+$ as $\bar{m}$ is fixed. We have $c = \delta_M - X_0 - \bar{m} - X_1\bar{m}^{\alpha}$. The second curve is obtained from relation (51) for $\dot{c} = 0$, $c = -\frac{a}{\bar{M}} + a'$ with $a = \pi\bar{m}^{\alpha}\left(\frac{r+\delta}{\epsilon B}\right)^{\alpha-1}$ and $a' = \delta_M + \frac{r+\delta(1-\epsilon)}{\epsilon} - \bar{m}$. The support of c is $\mathbb{R}_+^*$ and the derivatives are such that $\frac{dc}{d\bar{M}} = \frac{a}{\bar{M}^2}$ and $\frac{d^2c}{d\bar{M}^2} = -\frac{2a}{\bar{M}^{-3}}$ meaning that the function c is increasing and concave. Moreover we have $\lim_{\bar{M}\to\infty} c = a'$. The quantities

C_0, C_1, $\bar{M}_0$ and $\bar{M}_1$ in Figure (5) are computed as $C_0 = \delta_M + \frac{r+\delta(1-\epsilon)}{\epsilon} - \frac{r-\delta\epsilon-\rho}{\sigma_1} - \bar{m}_0 - \bar{m}_0^\alpha$, $C_1 = \delta_M + \frac{r+\delta(1-\epsilon)}{\epsilon} - \bar{m}_0$, $\bar{M}_0 = \frac{\pi\left(\frac{r+\delta}{\epsilon B}\right)^{\frac{\alpha-1}{\epsilon-1}} \bar{m}_0^\alpha}{\delta_M + \frac{r+\delta(1-\epsilon)}{\epsilon} - \bar{m}_0}$ and $\bar{M}_1 = \frac{\pi\left(\frac{r+\delta}{\epsilon B}\right)^{\frac{\alpha-1}{\epsilon-1}} \bar{m}_0^\alpha}{\frac{r-\delta\epsilon-\rho}{\sigma_1} - \bar{m}_0^\alpha \frac{1-\sigma_2}{\sigma_1} \pi\left(\frac{r+\delta}{\epsilon B}\right)^{\frac{\alpha-1}{\epsilon-1}}}$ where $\bar{m}_0$ is fixed for $\bar{m}$. iii) Figure (6): For $\dot{c} = 0$, the (first) curve is given by $c = \delta_M - X_0 - \bar{m} - X_1 \bar{m}^\alpha$. We have $\frac{dc}{d\bar{m}} = -1 - \alpha X_1 \bar{m}^{\alpha-1} > 0 \Leftrightarrow X_1 < 0 \Rightarrow \sigma_2 > 1$. Furthermore, $\frac{d^2c}{d\bar{m}^2} = -\alpha(\alpha-1)X_1\bar{m}^{\alpha-2}$ meaning concavity. So the function increases for $\bar{m} < \bar{m}_1$ and decreases when $\bar{m} > \bar{m}_1$, with $\bar{m}_1 = (-\alpha X_1)^{\frac{1}{1-\alpha}}$. The (second) curve for $\dot{\bar{m}} = 0$ is $c = -X_5 - \bar{m} - X_4 \bar{M}^{-1} \bar{m}^\alpha$. We have $\frac{dc}{d\bar{m}} = 1 - \alpha X_4 \bar{M}^{-1} \bar{m}^{\alpha-1}$ and $\frac{d^2c}{d\bar{m}^2} = -\alpha(\alpha-1)X_4\bar{M}^{-1}\bar{m}^{\alpha-2}$. It decreases and is convex. We have $C_0 = \delta_M + \frac{r+\delta(1-\epsilon)}{\epsilon} - \frac{r-\delta\epsilon-\rho}{\sigma_1}$, $C_1 = \delta + \frac{r+\delta(1-\epsilon)}{\epsilon}$ and $C_2 = \delta_M - X_0 - \bar{m}_0 - X_1(-\alpha X_1)^{\frac{\alpha}{1-\alpha}}$.

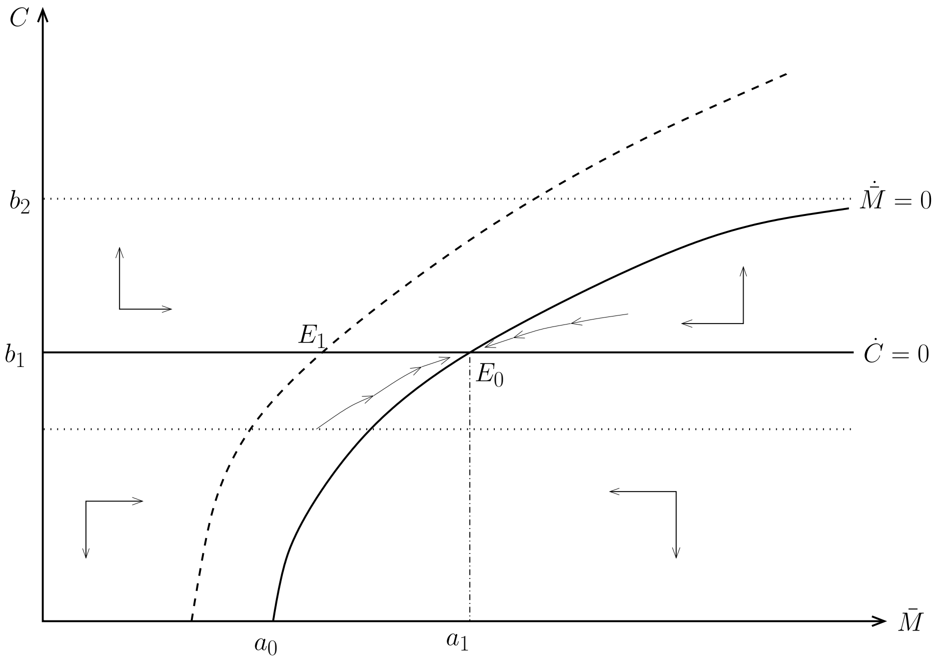

Figure 1: Phase diagram for separable preferences, plane $(C,\bar{M})$

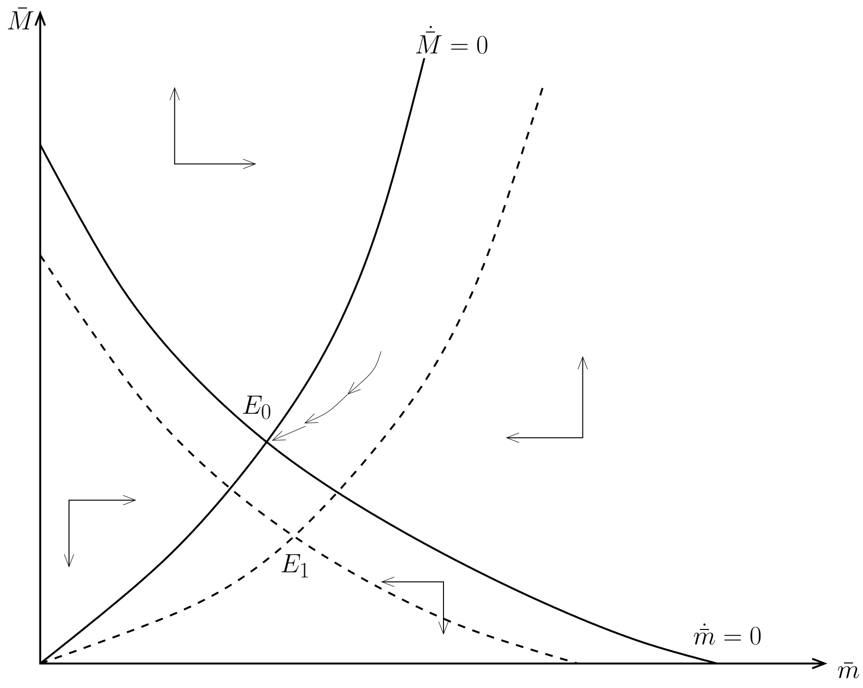

Figure 2: Phase diagram for separable preferences, plane ($\bar{M}$,$\bar{m}$)

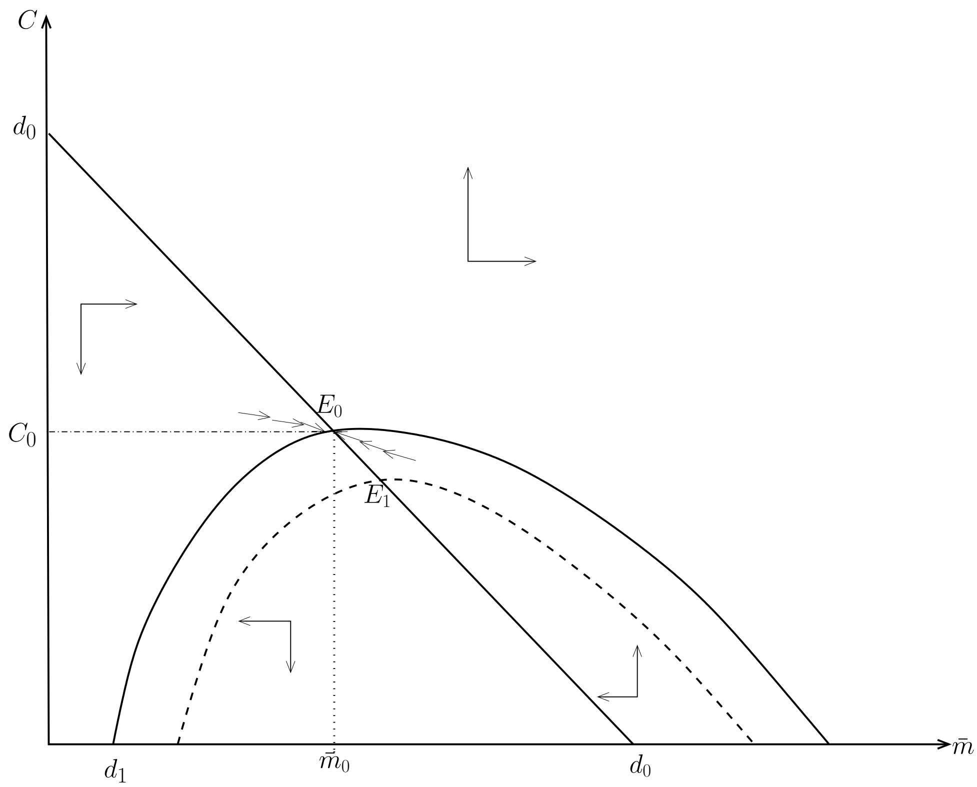

Figure 3: Phase diagram for separable preferences, plane (C, $\bar{m}$)

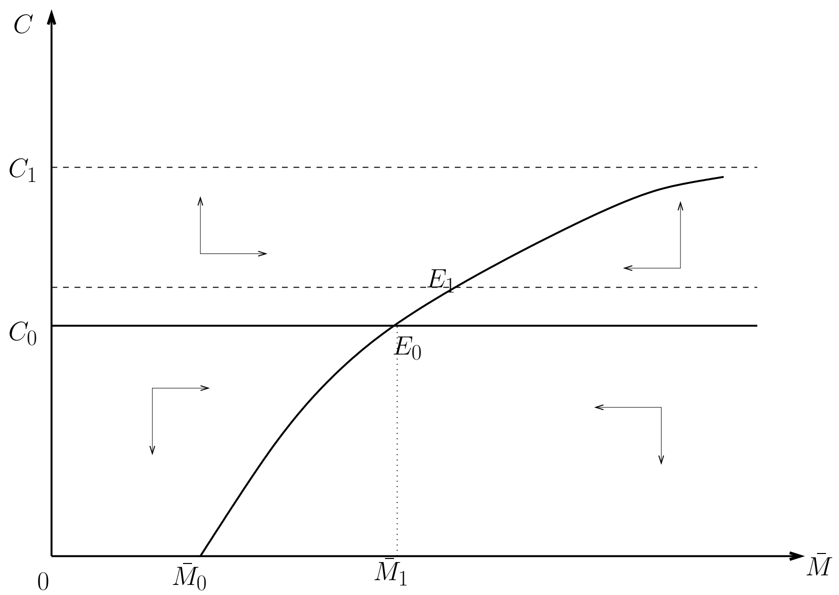

Figure 4: Phase diagram for non-separable preferences, plane (C,$\bar{M}$)

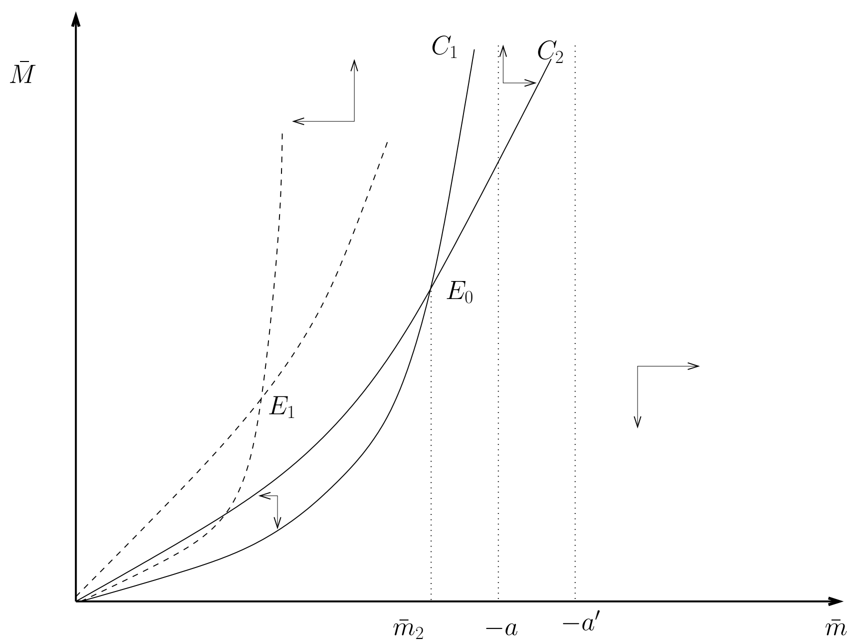

Figure 5: Phase diagram for non-separable preferences, plane ($\bar{M}$,$\bar{m}$)

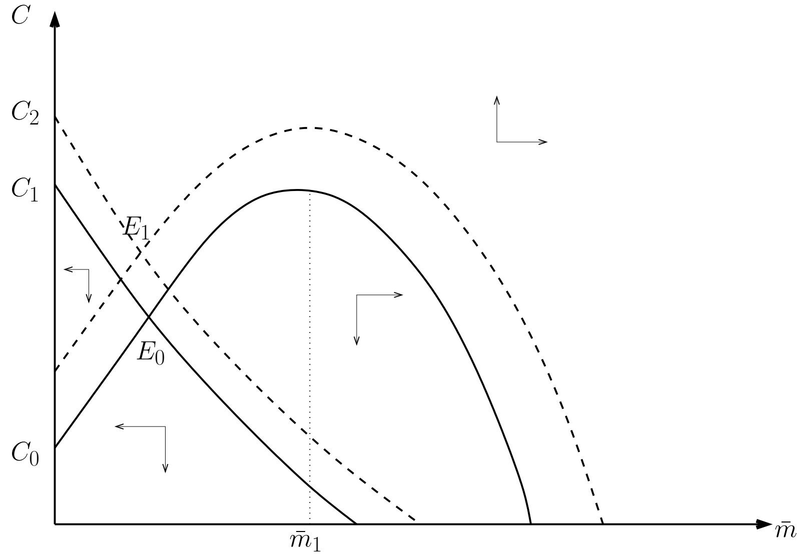

Figure 6: Phase diagram for non-separable preferences, plane (C,$\bar{m}$)

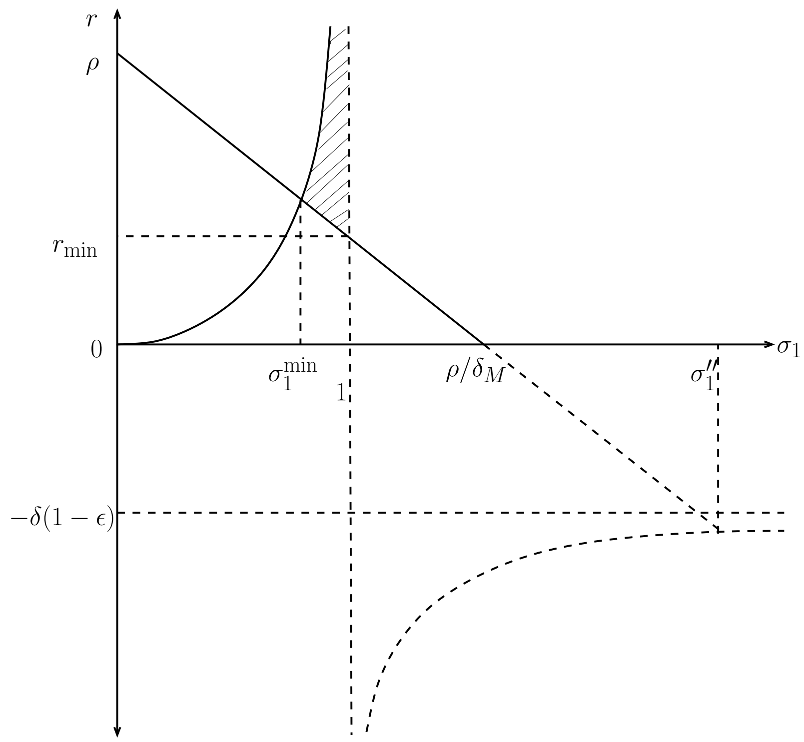

Figure 7: Geometrical illustration of parameters common support

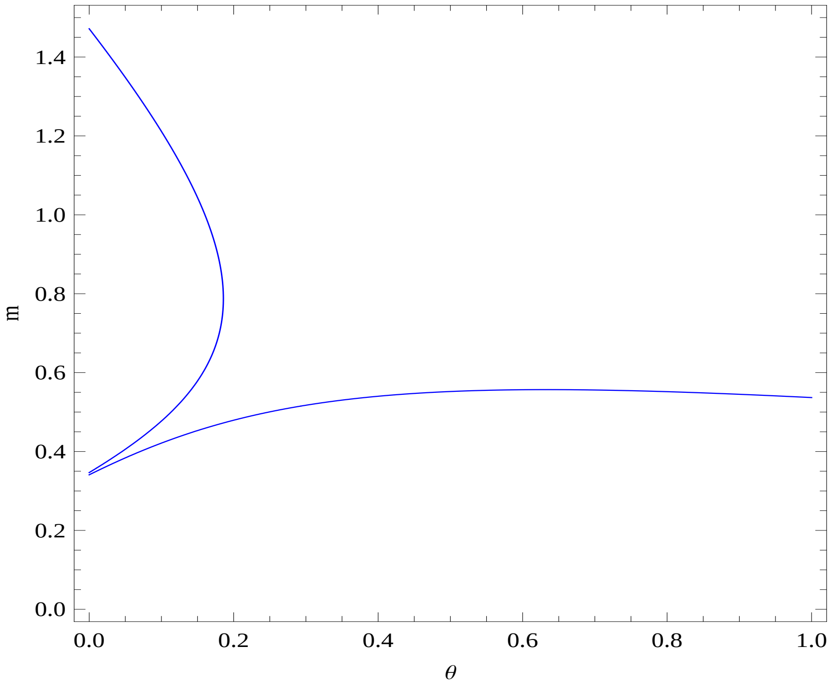

Figure 8: Bifurcation diagram on the plane ($\bar{m}$: [y-axis], $\theta := \delta_M$: [x-axis])

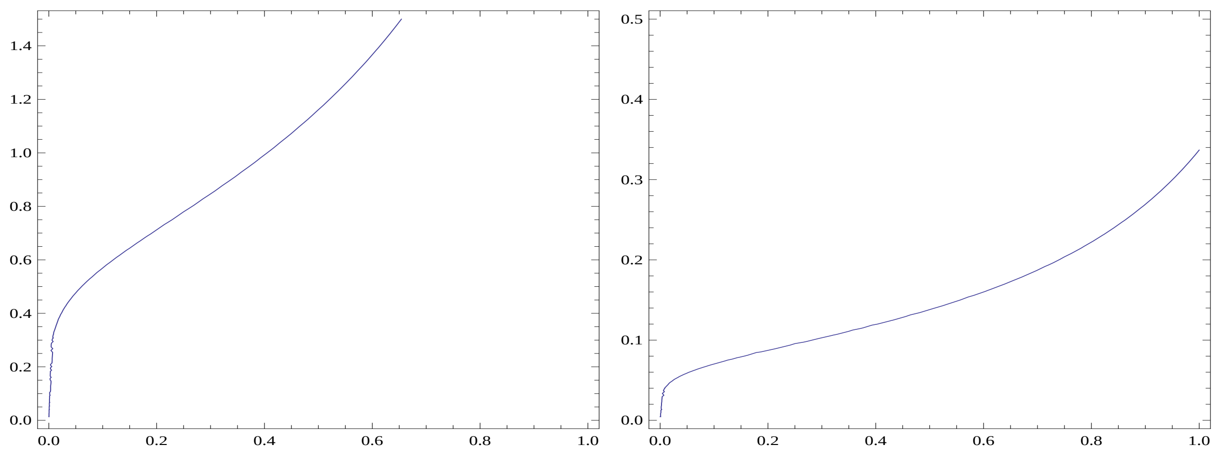

Figure 9: Weak health depreciation rate: Contour diagram on the plane ($\bar{M}$: [y-axis], $\bar{m}$: [x-axis])

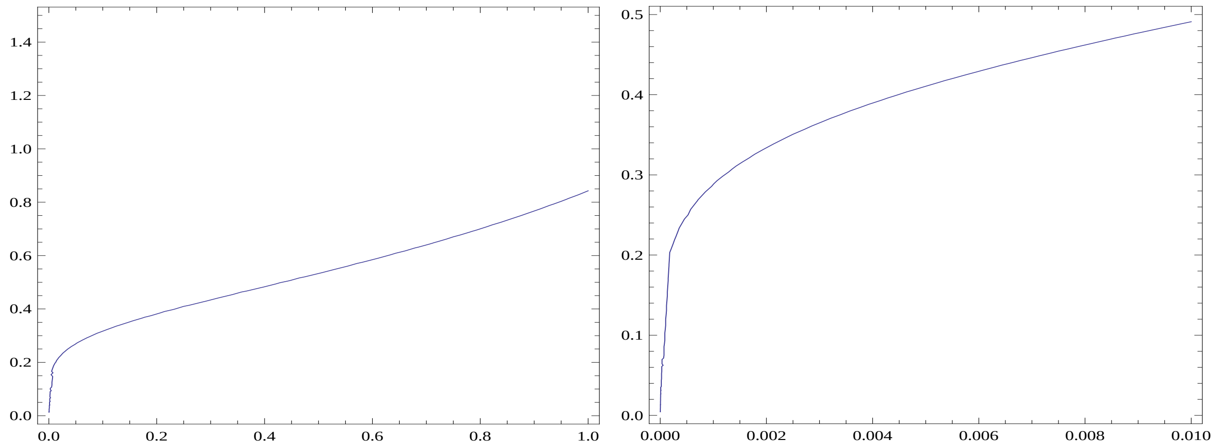

Figure 10: High health depreciation rate: Contour diagram on the plane ($\bar{M}$: [y-axis], $\bar{m}$: [x-axis])

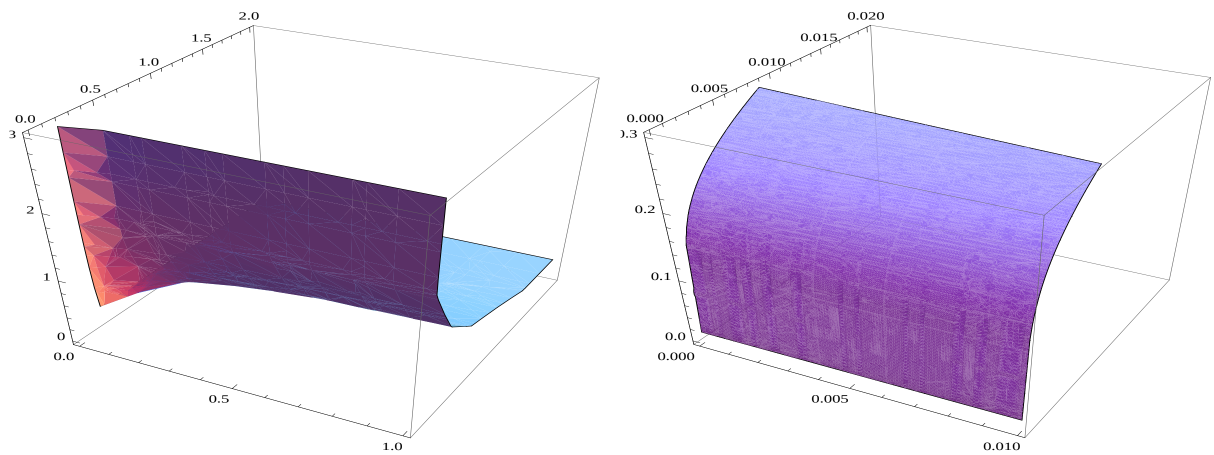

Figure 11: Weak health depreciation rate: 3 dimensional diagram (x-axis: consumption [C], y-axis: flow of investment in health capital [m], z-vertical axis: stock of health capital [M])

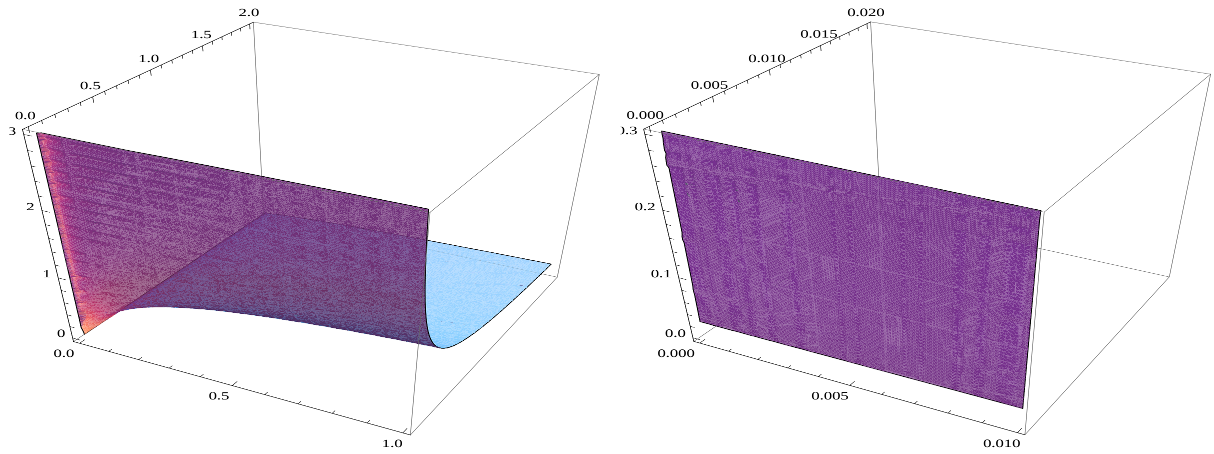

Figure 12: High health depreciation rate: 3 dimensional diagram (x-axis: consumption $[C]$, y-axis: flow of investment in health capital $[m]$, z-vertical axis: stock of health capital $[M]$)

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