MERIT-Infonomics Research Memorandum series

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2003-025

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Variety-robust axiomatic indices*

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October 2, 2003

Abstract

The price and quantity indices proposed in this paper exactly and unconditionally pass the product test, $E = PQ$. Because the indices are measured over single sets of products they can be used for the comparison of sets that do not contain the same types of products. Changes in product variety do not lead to biases in these indices. For homothetic functions, the indices can be used to decompose changes in productivity and the cost of living.

Shannon's (1948) entropy is shown to have properties that make it suitable as an index of product variety. This index of product variety is strongly related to the presented price and quantity indices.

Keywords: Index numbers, product variety, entropy
JEL classification: C43, E31, O47

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*I would like to thank participants to the SOM/TEG Conference 2002 in Groningen, especially Manuel Trajtenberg as well as Adriaan van Zon and Bart Verspagen for helpful comments.

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1 Introduction

Variety is the *raison d'être* of indices. Without product variety there would hardly be a need for the various indices devised to measure the quantity and price of sets of goods. Indeed, for a set of homogenous goods the ideal quantity index simply is the number of goods in the set, or some equivalent measure like their joint mass or volume. It is also straightforward to compute a representative price, even if for some reason these goods would be traded against different prices, by dividing total expenditure by the total quantity. This amounts to the same thing as a quantity-weighted arithmetic average of prices: $P = (\sum p_i x_i)/\sum x_j = \sum p_i(x_i/\sum x_j)$. ($p_i$ and x_i are the price and quantity of a type of good respectively; P is the price index.)

Drobisch (1871) suggested to use this method also for heterogeneous products. In a reaction to Drobisch, Laspeyres (1871) argued that simply dividing total expenditure through the number of goods does not yield a satisfactory price index when products are heterogeneous.

> Die Formel von Drobisch genügt nicht, sie kann ad absurdum führen, denn nach derselben kann sich eine durchschnittliche Preissteigerung oder Preissenkung ergeben, während alle einzelnen Waaren in Preise gleich geblieben sind. (Laspeyres, 1871, p. 308)

Laspeyres gives the following example to illustrate his criticism. Suppose the price of good A is 1 in both periods and the price of good B is 2, also in both periods. The quantity of A traded is 100 in period 0 and 1000 in period 1; the quantities of B are 100 and 20, respectively. Drobisch' index would then be 0.68, even though the prices of both goods remained constant.

Laspeyres' objection is not entirely justified. It does seem sensible to allow for changes in an index when the weights change, even when all prices remain constant. The problem with Drobisch' approach is a bit more subtle than Laspeyres suggested. Drobisch' method is unappealing because every good is considered to be equally important. As a consequence, a moderate change in the quantity of an unimportant type of good can have an enormous impact on the price index. This drawback can in principle be avoided by weighting the price of a good by, for example, its share in total expenditure. For some reason or another, however, this approach to indices has never been popular.

The primary purpose of an index being the comparison of different sets of goods, Laspeyres (1871) and Paassche (1874) devised price indices that measure the *difference* between sets, avoiding thereby the construction of indices that are *absolute* in the sense that they refer to just a single set.

Specifically, what Laspeyres and Paassche did was to compare the actual expenditure on one set with the hypothetical expenditure on the other set as if the quantities of the different types of goods would have been the same as in the first set. This 'relative' approach to the construction of indices has become something like a paradigm as virtually every economic index nowadays measures differences between sets and is not defined over a single set.

Two arbitrary decisions have to be made when computing a Laspeyres or Paassche index. First, a reference set, the set that is used for the computation of the weights, has to be chosen. Second, something should be done with types of goods that are present in one set of goods but not in other. For the first problem alternative indices have been devised such as the Fisher (first proposed by Bowley, 1899, p. 641) and Sato-Vartia indices.

The common way of dealing with the second problem is to exclude from the index the types of goods only present in one of the sets as if it is simply a problem of lacking data. This habit of ignoring that not every set of products contains the same types of goods has been criticized by Feenstra (1994), Feenstra and Shiells (1997), and Hausman (1997, 2003). The solutions they propose, however, are deeply rooted in economic theory and involve estimation of demand systems.

The indices proposed in this paper are not relative indices but are, like Drobisch' index, absolute meaning that they are computed for single sets of goods only. The difference in, for example, prices between two sets of goods can be determined simply by comparing the price index of one set with the price index of the other set. The important advantage of this 'absolute approach' is that the types of products do not have to be the same in both sets. Changes in product variety do not lead to biases in the indices (they are 'variety robust'). Furthermore, the proposed indices are 'axiomatic' in the sense that their properties are appealing outside of economic theory.[1] In particular, the indices pass the product test, $E = PQ$, exactly and unconditionally.

Besides the customary price and quantity indices also a complementary index of product variety is presented here, taking the form of the antilog of Shannon's (1948) entropy. Entropy has been employed as a measure of product variety before (e.g. Alexander, 1997), but a on a rather ad-hoc basis. In this paper, it is demonstrated (i) that, under certain conditions, Shannon's entropy can be used outside the domain of (probability) distributions without losing its properties, and (ii) that these properties make entropy suitable for

[1]Diewert (2001) distinguishes between axiomatic and economic indices. The important difference being that the latter type relies on the assumption of optimizing behavior, whereas the former type does not.

using it as an index of product variety.

The paper proceeds as follows. In section two, the indices and their axiomatic properties are discussed. Section three contains applications of the indices to economic theory, in particular to the measurement of productivity and the 'cost of living' albeit these applications are limited to homothetic functions. Additional results are presented for CES functions. The properties of the proposed variety index are the subject of section four. Concluding remarks can be found in section five.

2 Variety robust indices

Index number theory has traditionally sought to find the price and quantity indices that are consistent with expenditure. Perfect consistency would entail the following two identities.

$$E = PQ \tag{1a}$$

$$\hat{E} = \hat{P} + \hat{Q} \tag{1b}$$

Here E stands for total expenditure, P and Q are the price and quantity indices respectively. The hat denotes log differences. Contrary to conventional notation, indices are assumed to be 'absolute' (meaning that they are measures on a single set of goods), unless the context indicates otherwise.

To my knowledge, no indices apart from the ones proposed in this paper satisfy both conditions exactly and unconditionally.[2] Besides the fact that theorists focus on relative indices rather than absolute ones, the failure to derive indices that also satisfy the first condition may be attributable to the presumption that the price and quantity indices should ideally have the same algebraic form.

In order to see why this presumption is incorrect, consider first the simple case of symmetric product types. For all g product types $i \in G$ we have $x_i = \bar{x},\ p_i = \bar{p}$. Total expenditure is given by

$$E_G \equiv \sum_G p_i x_i = g\bar{p}\bar{x}.$$

The standard approach would then take $Q = g\bar{x}$ in order to get $E_G = PQ$. The aggregate quantity index Q is the product of a variety index and the average quantity across product types. This is exactly the reason why P and

[2] The Sato-Vartia and Fisher price and quantity indices have been shown to satisfy the second condition as long as the sets to be compared contain the same types of products (Sato 1976, Vartia 1974, Diewert 2003).

Q should *not* have the same functional form, and why ignoring this causes problems as soon as g starts to change.

It turns out to be that in the more general case where prices and quantities are asymmetrical, the antilog of Shannon's entropy can be used to replace g, while $\bar{p}$ and $\bar{x}$ can be substituted by the geometrical averages of the p_i and x_i. All three indices use the same weights.

Theorem 1 *Total expenditure can be decomposed into indices of variety, price, and quantity.*

$$E_G = NPX \tag{2a}$$

$$\ln N = \sum_G w_i \ln \frac{1}{w_i} \tag{2b}$$

$$\ln P = \sum_G w_i \ln p_i \tag{2c}$$

$$\ln X = \sum_G w_i \ln x_i \tag{2d}$$

$$w_i \equiv \frac{p_i x_i}{\sum_G p_j x_j} \tag{2e}$$

Proof.

$$\begin{aligned}
\ln (NPX) &= \sum_G w_i \left(\ln \frac{1}{w_i} + \ln p_i + \ln x_i \right) \\
&= \sum_G w_i \ln \frac{p_i x_i}{w_i} \\
&= \sum_G w_i \ln \frac{p_i x_i}{p_i x_i / \sum_G p_j x_j} \\
&= \ln \left(\sum_G p_j x_j \right) \sum_G w_i = \ln E_G
\end{aligned}$$

QED.

A generalized version of this theorem and its proof can be found in the appendix.

When we translate this result into the standard form, we get:

$$Q = NX \tag{3a}$$

$$\ln Q = \sum_G w_i \left(\ln \frac{x_i}{w_i} \right) = \sum_G w_i \left(\ln \frac{E_G}{p_i} \right). \tag{3b}$$

The index Q, which I will call the *aggregate* quantity index, clearly has a functional form that differs from that of P.

As mentioned before, quantity and price indices are usually defined in terms of the relative changes. The relative change of P and Q can be derived straightforwardly. (Superscripts identify time or country.)

$$\hat{P} = \sum_{G^0 \cup G^1} w_i^1 \ln p_i^1 - w_i^0 \ln p_i^0$$

$$\hat{Q} = \hat{N} + \hat{X} = \sum_{G^0 \cup G^1} w_i^1 \ln \left(\frac{x_i^1}{w_i^1} \right) - w_i^0 \ln \left(\frac{x_i^0}{w_i^0} \right)$$

$$w_i^t \equiv \frac{p_i^t x_i^t}{\sum_{G^t} p_j^t x_j^t}.$$

Diewert (2001) gives a survey of tests for price and quantity indices. I will discuss the most important of these tests with regard to the indices proposed above. First, the indices above perfectly satisfy the *product test*: $E = PQ$. This is a significant advantage over Fisher indices, which do only satisfy the product test in relative terms. Second, the *time reversal test* requiring $\hat{P}^{0,1} = -\hat{P}^{1,0}$ is also passed: $\ln(P^1/P^0) = -\ln(P^0/P^1)$. Similarly, the *circularity test*, which requires invariance between the chain and fixed-base methods, is satisfied:

$$\hat{P}^{0,2} = \ln \left(\frac{P^2}{P^0} \right) = \ln \left(\frac{P^2}{P^1} \frac{P^1}{P^0} \right) = \hat{P}^{0,1} + \hat{P}^{1,2}.$$

The implication of this property is that the choice of a base year or country is not influential so no arbitrary decisions have to be made when computing the indices.

Furthermore, the indices are all nonnegative and continuous in their variables. In particular, P is a monotonically increasing function in p_i and X in x_i. The weights (w_i) are homogenous of degree zero in prices, quantities, and expenditure.

$$w_i = \frac{\lambda p_i x_i}{\sum \lambda p_j x_j} = \frac{p_i x_i}{\sum p_j x_j}$$

Therefore, the variety index (N) is also homogenous of degree zero in prices, quantities, and expenditure. The price index is homogenous of degree one in prices, and homogenous of degree zero in quantities.

$$\prod (\lambda p_i)^{w_i} = \prod \lambda^{w_i} \prod p_i^{w_i} = \lambda P$$

In the same way, the quantity index (X) is homogenous of degree one in quantities, and homogenous of degree zero in prices. The same applies to the aggregate quantity index as $Q = NX$. Because the price and (aggregate) quantity indices are homogenous of degree one in prices and quantity respectively, a change in the unit of measurement does not affect the relative change of an index over time or otherwise. The indices thus satisfy the *commensurability test*.

The indices P and X are geometric averages, and therefore they satisfy *mean value tests*. The index Q satisfies the mean value test in the sense that it lies between $\min_i \{E/p_i\}$ and $\max_i \{E/p_i\}$, i.e. between the minimum and maximum quantity of goods that is attainable with the budget E. Naturally, the mean value tests also hold for relative changes in the indices.

The *factor reversal test* is not satisfied in the sense that P does not have the same functional form in the p_i as has Q in the x_i. However, as has been discussed above, the idea that P and Q should be of the same functional form is misguided. The factor reversal test does apply to N, P, and X.

The proposed indices satisfy all criteria mentioned above. Next, I will discuss some tests that the indices do not pass. First of all, the indices are incompatible with the *constant prices and constant quantities tests*,

$$P\left(\mathbf{p}, \mathbf{x}^1\right) / P\left(\mathbf{p}, \mathbf{x}^0\right) = \sum_G p_i x_i^1 / \sum_G p_i x_i^0$$
$$P\left(\mathbf{p}^1, \mathbf{x}\right) / P\left(\mathbf{p}^0, \mathbf{x}\right) = \sum_G p_i^1 x_i / \sum_G p_i^0 x_i.$$

This is a direct consequence of the fact that the proposed indices use separate weights for each set of goods. In my view, this is not a disadvantage of the indices. A price (quantity) index should be allowed to change if weights change, even when prices (quantities) remain unaltered and it would seem reasonable to allow for weights to differ between periods or across countries. Violation of the constant prices and constant quantities tests does not seem to be a serious offence.

The same applies to the *Paassche and Laspeyres bounding test.* A large change in the quantity of one of the products while all prices remain unchanged, will yield Paassche and Laspeyres price indices equal to one. However, the proposed price index will generally change as a result of this change in quantity.

The *additivity test* states that for some set of reference prices p_i^* that are equal for both sets of goods, it must hold that

$$\frac{Q^1}{Q^0} = \frac{\sum p_i^* x_i^1}{\sum p_i^* x_i^0}.$$

The additivity condition is not satisfied for the trivial reason that it implies that the price of a good is used as a weight, not the value of the good.

An issue that has gained attention relatively recently, concerns the bias on price indices induced by the introduction of new products and the extinction of old ones. Referring back to the introduction, modern indices of quantity and price measure a difference, usually the difference between two periods or two countries. These relative indices suffer from a fundamental flaw when (i) data on prices are required for the computation of the index and (ii) the set of products underlying the index is different in both periods of countries. The problem is simply that no price can be observed for non-traded goods. A product present in just one of two periods thus has to be excluded from the index that measures the change in, for example, prices between the two periods.

The problem is demonstrated easily with the Laspeyres price index.[3]

$$P_L = \sum p_i^1 x_i^0 / \sum p_j^0 x_j^0$$

This index reflects the change in prices between period 0 and 1 using period 0 quantities as weights. For goods that are not sold in period 0 no price is observed in that period. This need not be a problem if one is willing to assume that the prices of these goods are not infinitely large as the weights for these products are zero.[4]

Now consider the case of the disappearing products. Products only sold in period 0 have no observed price in 1 but, contrary to products only sold in period 1, they have positive weights. Apparently, the Paassche index is not robust to changes in the variety of products.

Similar arguments can be made for Laspeyres quantity indices and Paassche indices and, consequently, also for Fisher indices as the latter is a geometric average of Paassche and Laspeyres indices. As another example, take a general form of the Sato-Vartia index (Sato 1976).

$$P_{SV} = \prod_i \left(p_i^1 / p_i^0\right)^{\phi_i}$$

If the price of a product i is zero in period 1, the index either excludes the product or the index will be infinitely large, depending on whether $\phi_i \leq 0$ or not.

[3]See also the simulation results in appendix B.

[4]Assuming a price to be infinitely large is not as odd as it might appear. In 1871 a walk on the Moon could not have be bought with all the money in the world. It is the invention of a good that makes its price finite.

Hausman (1997) has proposed a solution that solves this problem of 'missing prices' for cost of living indices. Essentially, Hausman computes the shadow price for goods that are not sold during one of the periods by estimating a demand system. From a practical point of view this is a rather awkward solution as it involves the estimation of demand systems for each new or disappearing product, something that is difficult to automate reliably considering Hausman's efforts to estimate a demand system for just a single new product, like Apple-Cinnamon Cheerios.

Feenstra (1994) has developed a price index capable of handling changing product sets that is easier to estimate but this index is only defined for preferences or production technologies that can be represented by a CES function.

The results of numerical simulations comparing P and Q with Laspeyres, Paassche, and Fisher indices can be found in appendix B. Here, N is also compared with the Feenstra variety index.

3 Applications to economic theory

3.1 Productivity measurement

From an axiomatic point of view, a measure of total factor productivity growth (TFPG) that is based on the indices of the previous section is easily constructed: simply compare the ratio of the quantity index of the inputs, Q_x, to that of the outputs, Q_y, in one year or country to that of another one.

$$TFPG = \ln\left(Q_y^1/Q_x^1\right) - \ln\left(Q_y^0/Q_x^0\right)$$

It is more complicated to show that this TFPG-index is consistent with economic theory. In fact, I will limit my discussion to homothetic production functions with many inputs and just one output. Consider a continuously differentiable homothetic production function of the form

$$y = h\left(f\left(\mathbf{x}\right)\right), \tag{4}$$

where y is the quantity of output produced, $\mathbf{x}$ is a vector of input quantities, f is a homogenous function of degree one, and h is a continuous, monotonically increasing function of f.[5]

Profit maximization yields a price that is a markup over marginal costs,

$$\max_{x_i}\left\{p_y\left(y\right)y - \sum p_i x_i\right\} \Rightarrow p_i = \left(1+\varepsilon_{p_y y}\right)h' f_i p_y$$

[5] It is shown in the appendix that the results in this section also apply to the CRESH function (Hanoch 1971), which is implicitely homothetic.

where $\varepsilon_{p_y y}$ is the elasticity of demand.

Cost shares are given by

$$w_i = \frac{p_i x_i}{\sum p_j x_j} = \frac{\left(1+\varepsilon_{p_y y}\right) h' f_i p_y x_i}{\sum \left(1+\varepsilon_{p_y y}\right) h' f_j p_y x_j} = \frac{f_i x_i}{\sum f_j x_j}. \tag{5}$$

Euler's theorem allows us to write $f(\mathbf{x}) = \sum f_i x_i$. Applying the theorem provided in the appendix to the right hand side of this expression, f can be decomposed into a variety index N, an index of the partial derivatives F', and the average quantity index of the inputs X.

$$f(\mathbf{x}) = NF'X$$
$$F' = \prod f_i^{w_i}, \qquad w_i = \frac{f_i x_i}{\sum f_j x_j}$$

The assumption of homotheticity assures that the weights used in the indices equal the cost shares of the corresponding products.

Total factor productivity growth is the log change in y/Q and can be decomposed as follows.

$$\begin{aligned} \mathrm{d}\ln\left(\frac{y}{Q}\right) &= \mathrm{d}\ln h\left(NF'X\right) - \mathrm{d}\ln Q \\ &= \varepsilon_{hf}\left(\mathrm{d}\ln N + \mathrm{d}\ln F' + \mathrm{d}\ln X\right) - \mathrm{d}\ln Q \\ &= \left(\varepsilon_{hf} - 1\right)\mathrm{d}\ln Q + \varepsilon_{hf}\mathrm{d}\ln F' \end{aligned}$$

Total factor productivity growth depends on the quantity of inputs used (a scale effect) and on the index of partial derivatives of $f(\mathbf{x})$ (a technology effect). These two effects are moderated by ε_{hf}, the 'elasticity of scale'. Of course, this decomposition only is exact for marginal changes in y and $\mathbf{x}$.

Usually we are not interested in marginal but in substantial differences in productivity. Allowing for substantial differences the decomposition of TFPG is still exact if h is homogeneous rather than homothetic. For homogenous functions of degree r, Euler's theorem states that $h = \frac{1}{r}\sum h_i x_i$. Decomposing h yields $\frac{1}{r}NH'X$ with the weights $h_i x_i / \sum h_j x_j$ equaling expenditure shares. TFPG allowing for substantial differences now can be stated as

$$\Delta\ln\left(\frac{y}{Q}\right) = \Delta\ln H'. \tag{6}$$

The main disadvantage of this measure of TFPG is that its 'economic' properties for production processes with multiple outputs are unknown. If one does not want to rely on axiomatic properties alone, Malmquist indices are still to be preferred over $\Delta\ln\left(Q_y/Q_x\right)$ (Caves, Christensen, and Diewert 1982).

3.2 CES functions

If we are willing to impose even more structure on the production function and require it to have constant elasticity of substitution between inputs (CES), TFPG can be decomposed even further. The production function given by

$$y = h(\mathbf{x}) = \left(\sum \theta_i x_i^\alpha\right)^{\frac{r}{\alpha}},$$

allows for y to be expressed in terms of N and X using the theorem in the appendix.

$$y^{\frac{\alpha}{r}} = \sum \theta_i x_i^\alpha = N\Theta X^\alpha \Rightarrow y = N^{\frac{r}{\alpha}} \Theta^{\frac{r}{\alpha}} X^r$$
$$w_i = \frac{\theta_i x_i^\alpha}{\sum \theta_j x_j^\alpha}$$

Here, Θ, a 'quality-index', is the weighted geometric average of the θ_i. It remains to be checked whether these weights equal the cost shares. Profit maximization implies that the price of good i equals $\left(1+\varepsilon_{p_y y}\right) r y^{1-\frac{\alpha}{r}} \theta_i x_i^{\alpha-1} p_y$. Applying this to the cost shares confirms that the weights are indeed correct.

$$w_i = \frac{p_i x_i}{\sum p_j x_j} = \frac{\left(1+\varepsilon_{p_y y}\right) r y^{1-\frac{\alpha}{r}} \theta_i x_i^\alpha p_y}{\sum \left(1+\varepsilon_{p_y y}\right) r y^{1-\frac{\alpha}{r}} \theta_j x_j^\alpha p_y} = \frac{\theta_i x_i^\alpha}{\sum \theta_j x_j^\alpha}$$

Using $Q = NX$ an expression for TFPG can readily be obtained:

$$\Delta \ln\left(\frac{y}{Q}\right) = \Delta \ln\left(N^{\frac{(1-\alpha)r}{\alpha}} \Theta^{\frac{r}{\alpha}} Q^{r-1}\right)$$
$$= (r-1)\Delta \ln Q + \frac{(1-\alpha)r}{\alpha}\Delta \ln N + \frac{r}{\alpha}\Delta \ln \Theta$$

The first term is the scale effect, the second is the effect due to a change in product variety, and the last term is the effect attributable to changes in quality.

Although the θ_i are usually unobserved, it might be possible to compute them. Recall that the cost shares are given by $w_i = \theta_i x_i^\alpha / \sum \theta_j x_j^\alpha = \theta_i x_i^\alpha / y^{\frac{\alpha}{r}}$. If w_i, x_i, and y are observed and α and r can be estimated, $\theta_i = w_i x_i^{-\alpha} y^{\frac{\alpha}{r}}$ can be computed and consequently so can Θ. The imposition of a CES production structure allows for a practical decomposition of productivity growth into the effects of scale, product variety, and quality.

Next, I will briefly discuss two special cases that are popular with researchers working on models of economic growth, being $y = \sum \theta_i x_i^\alpha$ and

$y = (\sum \theta_i x_i^\alpha)^{\frac{1}{\alpha}}$. The first case corresponds with $r = \alpha$, the second with $r = 1$. It is interesting to see what consequences these assumptions on r have for TFPG.

$$\begin{aligned} r = \alpha \;\Rightarrow\; \Delta \ln \left(\frac{y}{Q}\right) &= (\alpha - 1)\, \Delta \ln X + \Delta \ln \Theta \\ r = 1 \;\Rightarrow\; \Delta \ln \left(\frac{y}{Q}\right) &= \frac{1-\alpha}{\alpha} \Delta \ln N + \frac{1}{\alpha} \Delta \ln \Theta \end{aligned}$$

When $r = \alpha$ TFPG is negatively affected by the average scale of production, and positively affected by increases in quality. Changes in product variety do not matter for TFPG. When $r = 1$ both increases in product variety and quality affect TFPG positively while the average scale of production is not influential.

In the seminal paper by Romer (1990) r is taken to equal α. TFPG is driven by reducing X thereby escaping decreasing marginal returns to intermediates. Increasing product variety merely facilitates this mechanism; without the introduction of new products the total amount of goods would have to decline in order to achieve positive TFPG.

The specification $r = 1$ has been used by e.g. Grossman and Helpman (1991). Now X does not matter anymore for TFPG (constant returns to scale) and changes in product variety directly affect TFPG. New products cause older products to perform better. Clearly, a seemingly harmless assumption on r yields a completely different reason for TFPG.

Feenstra, Markusen, and Zeile (1992) and Feenstra et al (1999) suggest a method for measuring TFPG driven by the introduction of new goods. Their index, which I will call the Feenstra variety index, requires a CES function with $r = 1$.

$$TFPG_F = \frac{1-\alpha}{\alpha} \ln \left(\frac{\sum_{i \in G^0} p_i^0 x_i^0 / \sum_{i \in (G^0 \cap G^1)} p_i^0 x_i^0}{\sum_{i \in G^1} p_i^1 x_i^1 / \sum_{i \in (G^0 \cap G^1)} p_i^1 x_i^1} \right)$$

For symmetric goods, the Feenstra index reduces to the same formula for TFPG as $\frac{1-\alpha}{\alpha} \Delta \ln N$.

$$\frac{1-\alpha}{\alpha} \ln \left(\frac{g^0 p^0 x^0 / (g^* p^0 x^0)}{g^1 p^1 x^1 / (g^* p^1 x^1)} \right) = \frac{1-\alpha}{\alpha} \ln \left(\frac{g^0}{g^1} \right)$$

g^* is the number of product types in the set $G^0 \cap G^1$. A peculiar property of the Feenstra index is that changes in the prices or quantities of product types present in both G^0 and G^1 only affect the index if $G^0 \neq G^1$. Products types weighted only when the set of product types changes.

A disadvantage of the Feenstra index is that it does not allow for the separation of the effects of changing product sets from the effects of quality changes. The TFPG index proposed in this section allows for both scale and quality effects and does not make any assumptions not necessary for the Feenstra index. On these grounds, the proposed TFPG index is to be preferred over the Feenstra index.

3.3 Cost of living indices

Cost of living indices (COLI) differ from normal price indices in that they refer to the effect of a change in prices rather than to the change in prices itself. In principle, the cost of living can change for two reasons. First, the prices of consumer goods may change. Second, consumers may appreciate their basket of goods differently than before. This second component requires a COLI to be rooted into economic theory. Below I will show how the indices proposed in this paper can be used for the construction of a COLI based on the homothetic utility function

$$u = h\left(f\left(\mathbf{x}\right)\right), \qquad (7)$$

where u is utility, $\mathbf{x}$ is a vector of quantities of consumer goods, f is a homogenous function of degree one, and h is a continuous, monotonically increasing function of f.

The consumer maximizes utility subject to the budget B.

$$\max_{x_i}\left\{u\right\} \text{ s.t. } B = \sum p_i x_i \Rightarrow \max_{x_i}\left\{h\left(f\left(\mathbf{x}\right)\right) + \lambda\left(B - \sum p_i x_i\right)\right\}$$

From the first order condition follows that prices are given by $p_i = \lambda^{-1}h'f_i$, with λ being the Lagrange multiplier. Consequently, budget shares can be expressed as in 5.

$$w_i = \frac{p_i x_i}{\sum p_j x_j} = \frac{\lambda^{-1}h'f_i x_i}{\sum_j \lambda^{-1}h'f_j x_j} = \frac{f_i x_i}{\sum_j f_j x_j} \qquad (8)$$

Applying Euler's theorem and the theorem from the appendix, the utility function can be rewritten as $u = h\left(F'Q\right)$. For utility functions that are homogenous of degree r this becomes $u = \frac{1}{r}H'Q$ (see subsection 3.1).

A COLI is defined as the unit cost of utility and is decomposed as

$$c = E/u = \frac{PQ}{h\left(F'Q\right)}$$
$$\mathrm{d}\ln c = \mathrm{d}\ln P - \left(\varepsilon_{hf} - 1\right)\mathrm{d}\ln Q - \varepsilon_{hf}\mathrm{d}\ln F'$$

for the homothetic case, and as

$$
\begin{aligned}
c = E/u &= \frac{PQ}{\frac{1}{r}H'Q} = rP/H' \\
\Delta \ln c &= \Delta \ln P - \Delta \ln H'
\end{aligned}
$$

for the homogenous case. Changes in the cost of living are caused by a change in the price index of consumption goods and by a change in the 'efficiency' of consumption.

The equivalence of these results with COLI's based on CES-preferences is easily demonstrated. As has been demonstrated in subsection 3.1 the CES function $u = (\sum \theta_i x_i^\alpha)^{\frac{r}{\alpha}}$ can be written as $N^{\frac{r}{\alpha}}\Theta^{\frac{r}{\alpha}}X^r$. The COLI belonging to this function is easy to derive.

$$
\begin{aligned}
c = E/u &= \frac{NPX}{N^{\frac{r}{\alpha}}\Theta^{\frac{r}{\alpha}}X^r} = N^{1-\frac{r}{\alpha}}\Theta^{\frac{-r}{\alpha}}PX^{1-r} \qquad (9) \\
\Delta \ln c &= \left(1 - \frac{r}{\alpha}\right)\Delta \ln N - \frac{r}{\alpha}\Delta \ln \Theta + (1-r)\,\Delta \ln X + \Delta \ln P
\end{aligned}
$$

The cost of living decline with increases in product variety (if $r > \alpha$), product quality, and the scale of production (if $r > 1$), but it rises with increases the price index of consumption goods.

Normally a CES COLI is derived differently. Standard manipulation yields the following index[6]

$$
c = u^{1/r-1}\left(\sum \theta_i^{\frac{1}{1-\alpha}} p_i^{\frac{\alpha}{\alpha-1}}\right)^{\frac{\alpha-1}{\alpha}}.
$$

This standard index can be written as a function of weighted geometric means:[7]

[6] Usually this index is computed for $r = 1$.

[7] The weights can be shown to equal expenditure shares. The first order conditions imply that $p_i x_i = \theta_i^{\frac{1}{1-\alpha}} p_i^{\frac{\alpha}{\alpha-1}} c^{\frac{1}{1-\alpha}} u^{\frac{1-\alpha/r}{1-\alpha}}$, and thus expenditure shares become

$$
w_i = \frac{\theta_i^{\frac{1}{1-\alpha}} p_i^{\frac{\alpha}{\alpha-1}} c^{\frac{1}{1-\alpha}} u^{\frac{1-\alpha/r}{1-\alpha}}}{\sum \theta_j^{\frac{1}{1-\alpha}} p_j^{\frac{\alpha}{\alpha-1}} c^{\frac{1}{1-\alpha}} u^{\frac{1-\alpha/r}{1-\alpha}}} = \frac{\theta_i^{\frac{1}{1-\alpha}} p_i^{\frac{\alpha}{\alpha-1}}}{\sum \theta_j^{\frac{1}{1-\alpha}} p_j^{\frac{\alpha}{\alpha-1}}}.
$$

$$
\begin{aligned}
c &= u^{1/r-1} N^{\frac{\alpha-1}{\alpha}} \tilde{\Theta}^{\frac{\alpha-1}{\alpha}} \tilde{P}^{\frac{\alpha-1}{\alpha}} \\
\tilde{\Theta} &= \prod \theta_i^{\frac{1}{1-\alpha} w_i} = \left(\prod \theta_i^{w_i}\right)^{\frac{1}{1-\alpha}} = \Theta^{\frac{1}{1-\alpha}} \\
\tilde{P} &= \prod p_i^{\frac{\alpha}{\alpha-1} w_i} = \left(\prod p_i^{w_i}\right)^{\frac{\alpha}{\alpha-1}} = P^{\frac{\alpha}{\alpha-1}} \\
w_i &= \frac{\theta_i^{\frac{1}{1-\alpha}} p_i^{\frac{\alpha}{\alpha-1}}}{\sum \theta_j^{\frac{1}{1-\alpha}} p_j^{\frac{\alpha}{\alpha-1}}}
\end{aligned}
$$

Substitution of $\tilde{\Theta}$ and $\tilde{P}$, and subsequently of u, returns the form of the COLI in 9.

$$
\begin{aligned}
c &= u^{1/r-1} N^{\frac{\alpha-1}{\alpha}} \Theta^{\frac{-1}{\alpha}} P \\
&= N^{1-\frac{r}{\alpha}} \Theta^{\frac{-r}{\alpha}} X^{1-r} P
\end{aligned}
$$

The Sato-Vartia price index is an exact COLI for CES-preferences provided that three conditions have been met. First, it assumes that the parameters θ_i are the same for both sets of goods. Second, it is only exact if both sets of goods contain the same product types. Third, it requires $r = 1$, or constant returns to scale.[8]

Feenstra (1994) ‘upgraded’ the Sato-Vartia index in order to let it meet the first and second criterion. Feenstra’s COLI does not allow for the separation of effects due to new or disappearing products and effects due to changes in preferences or quality. Feenstra’s results only apply for CES-functions with $r = 1$.

4 Entropy and product variety

For many purposes, the number of products is a poor index of product variety because each product is assigned the same weight. For example, ‘Crude oil’ would have the same contribution to the variety index as ‘Turkeys: whole: frozen, not cut in pieces’ would have. This cannot be desirable. Product types should be weighted according to their importance relative to other product types. The measure on which these weights should be based is dependent on the variable to which variety is to be related. For example,

[8]The COLI devised by Lloyd (1975) and Moulton (1996) is also exact under these conditions but it does not satisfy the relative product rule in the general case like the Sato-Vartia index does.

when investigating the relation between input variety and total output, each input type should be weighted in terms of its contribution to output.

Even with the appropriate weights in hand, it is not straightforward to measure variety. For example, simply summing the weights always yields one. Alternatively, summing the weights relative to the average weight simply returns the number of product types. It turns out to be that Shannon's entropy provides a convenient way of 'summing' the weights in a meaningful manner.[9] In this section it is demonstrated that, when certain conditions are being met, Shannon's entropy can be used outside the domain of (probability) distributions without losing its properties.

Suppose each product is assigned a weight (w) that depends on how important the product is compared to the other products as indicated by a measure u. It is desirable for an index of product variety (H) to have the following properties (these properties are identical to those proposed by Shannon for entropy):

1. H is continuous in all w_i.
2. If $w_i = \frac{1}{n}$ for all i, then H is monotonically increasing in n.
3. Total variety is the variety of subsets plus the weighted sum of variety within each subset. $H(N) = H(G) + \sum_{j \in G} w_j H(N_j)$ where N is the set of all product types, N_j is a subset of N such that $\cup_{j \in G} N_j = N$ and $N_j \cap N_k = \emptyset$ for all $j \neq k$, and G is the collection of subsets. I.e. if a product type is classified into subset N_j it cannot belong to another subset N_k of N.

The first property ensures that when an infinitesimal amount of a product is produced that did not exist previously, the variety index will not make a discrete jump but will change marginally. The second property implies equivalence with the most basic variety-index possible, (i.e. the number of products) once equal weights are assigned to products.

The third property is desirable because it relates the entropy between the entire set of products and its subsets. Property three ensures that the total entropy at a higher level of aggregation is lower than the total entropy at a more detailed level of aggregation, even when only one subset is disaggregated.

[9] It was Henry Theil (1965, 1967) who introduced the concept of entropy to economics, taking it from information theory where Shannon (1948, theorem 2) had previously developed what turned out to be a particularly popular type of entropy. Since Theil, entropy has been applied in economics with various objectives, including the measurement of income inequality and market concentration.

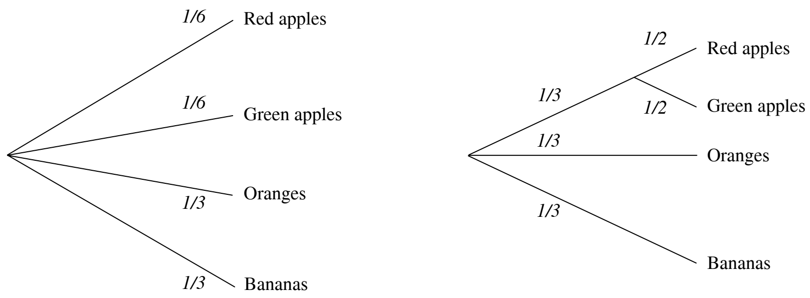


Figure 1: Entropy and product classification

In order to clarify this last point consider the following example. Suppose the set of products 'fruit' contains 'oranges', 'apples', and 'bananas'. Each type of fruit gets the weight 1/3. The subset 'apples' in turn consists of 'red apples' and 'green apples', each type of apple being weighted equally. Property three allows for the decomposition of entropy of the entire set of products into the entropies of subsets (see figure). If the subset 'apple' would not be disaggregated, total variety would be $H(1/3, 1/3, 1/3)$. The difference between total variety with two types of apples and total variety without different types of apples equals the contribution to total variety of variety in 'apples':

$$H(1/6, 1/6, 1/3, 1/3) - H(1/3, 1/3, 1/3) = \frac{1}{3} H(1/2, 1/2).$$

Theorem 2 *The only function H that possesses all three properties stated above is*

$$H(N) = -K \sum_{i \in N} w_i \ln w_i \tag{10a}$$

$$w_i \equiv \frac{u_i}{\sum_{j \in N} u_j} \tag{10b}$$

where K is an arbitrary constant, and u is a measure along which the elements of N (the product types) can be compared such that

$$u_N = \sum_{j \in N} u_j. \tag{10c}$$

Proof (similar to Shannon, 1948). Starting point is the situation in which every single product is considered a product type on its own. Note that this is the most detailed level at which products can be classified. All n products have the same weight: $w_i = 1/n$ for all i. In this symmetric case the variety index only depends on n and the notation can be simplified into $H(N) = A(n)$. Suppose that all products are classified into s broad subsets and that all these subsets are again subdivided into s subsets and so on until the number of times sets are subdivided is m. After m steps, the total number of subsets at the most detailed level of disaggregation will be s^m.

By property three this implies that $A(s^m)$ should satisfy

$$\begin{aligned} A(s^m) &= A(s) + s\frac{1}{s}A(s) + s^2\frac{1}{s^2}A(s) + .. + s^{m-1}\frac{1}{s^{m-1}}A(s) \\ &= mA(s). \end{aligned}$$

In the same way, if products are n times subdivided into t subsets per set, the total number of subsets at the most detailed level is t^n. From property three follows again that $A(t^n) = nA(t)$.

For any value of n, s, and t, an m can be found such that

$$s^m \leq t^n < s^{m+1}.$$

After taking logarithms and dividing by $n \ln s$ we get

$$\frac{m}{n} \leq \frac{\ln t}{\ln s} < \frac{m}{n} + \frac{1}{n}.$$

Rearranging yields

$$\begin{array}{l} 0 \leq \frac{\ln t}{\ln s} - \frac{m}{n} < \frac{1}{n} \\ \frac{m}{n} - \frac{\ln t}{\ln s} \leq 0 < \frac{1}{n} \end{array} \Rightarrow \left| \frac{\ln t}{\ln s} - \frac{m}{n} \right| < \varepsilon.$$

As n can take on any value, ε can be taken to be arbitrarily close to zero – provided that the number of products is large enough.

Property two implies that

$$\begin{aligned} A(s^m) \leq A(t^n) &< A\left(s^{m+1}\right) \\ mA(s) \leq nA(t) &< (m+1)A(s) \\ \frac{m}{n} \leq \frac{A(t)}{A(s)} &< \frac{m}{n} + \frac{1}{n} \\ \left| \frac{m}{n} - \frac{A(t)}{A(s)} \right| &< \varepsilon, \end{aligned}$$

and thus

$$\begin{array}{l} \frac{A(t)}{A(s)} - \frac{m}{n} + \frac{m}{n} - \frac{\ln t}{\ln s} < 2\varepsilon \\ \frac{m}{n} - \frac{A(t)}{A(s)} + \frac{\ln t}{\ln s} - \frac{m}{n} < 2\varepsilon \end{array} \rightarrow \left| \frac{A\left(t\right)}{A\left(s\right)} - \frac{\ln t}{\ln s} \right| < 2\varepsilon.$$

For ε arbitrarily close to zero, $A\left(t\right)$ should be of the form $K \ln t$. The constant K should be positive, otherwise H would not be increasing in n. The function $H\left(N\right) = K \ln n$ possesses all three properties for the symmetric case.

Now we turn to the asymmetric case. Suppose that all products are subdivided into groups. Each of the g groups consists of n_i products; the total number of (symmetric) products therefore is $\sum_G n_i$ with G being the set of groups. Then using property three the entropy of all products (regardless of their group) can be split up into the entropy among product groups and the weighted sum of the entropy within each product group.

$$\begin{aligned} A\left(n\right) &= H\left(G\right) + \sum_G w_i A\left(n_i\right) \\ w_i &\equiv \frac{n_i}{\sum_G n_j} \end{aligned}$$

Note that the entropy of all products and the entropy within each group are based on products that are of equal weight. Inserting of $A\left(n\right) = K \ln n$ and rearranging gives an expression for the entropy of product groups.

$$K \ln \left(\sum_{j \in G} n_j \right) = H\left(G\right) + K \sum_{i \in G} w_i \ln n_i$$

$$\begin{aligned} H\left(G\right) &= K \sum_{i \in G} w_i \ln \left(\sum_{j \in G} n_j \right) - K \sum_{i \in G} w_i \ln n_i \\ &= -K \sum_G w_i \ln w_i. \end{aligned}$$

This is Shannon's original definition of entropy. Now it remains to be shown that all three properties apply to a broad class of weights, not just to the proportion of the number of products.

Usually, it does not make sense to assume equal weights for products that differ substantially from each other. Suppose that there exists a measure u that allows for the comparison of different products. Product i possesses the equivalent of u_i (virtual) units of equal weight and u_N is the amount of virtual units measured over the set of products N. The entropy of these virtual units is arbitrary as any scale can be assigned to the measure u. This, however, does not prevent us from finding the entropy of the set of products.

$$A\left(u_N\right) = H\left(N\right) + \sum_N w_i A\left(u_i\right)$$
$$H\left(N\right) = K \sum_N w_i \ln \frac{u_N}{u_i}$$

In the symmetric case this would be

$$A\left(n\right) = K \ln \frac{u_N}{\bar{u}}.$$

As $A\left(n\right) = K \ln n$, this statement will not hold unless $u_N = n\bar{u}$. Returning to asymmetry, assume that u_M can be subdivided into g groups of measure $n_i\bar{u}$. Again, property three defines the relation between the entropy of u_N and its subgroups.

$$A\left(n\bar{u}\right) = H\left(G\right) + \sum_G \frac{n_i\bar{u}}{n\bar{u}} A\left(n_i\bar{u}\right)$$
$$H\left(G\right) = K \sum_G \frac{n_i}{n} \ln \frac{n}{n_i}$$

This last expression is identical to Shannon's entropy. Therefore, a sufficient requirement for the measure u to be preserving the properties of Shannon's entropy is that

$$u_N = \sum_N u_i.$$

As opposed to the scale of u, the scale of the weights is not arbitrary at all and thus the entropy of products of different types can be computed as long as a suitable measure can be found for the comparison of products.

The entropy of all single products is not a suitable measure for product variety. Product variety refers to the number of product types rather than the number of products itself. As a final step we therefore take the entropy of product groups rather than that of products.

$$K \ln \sum_{k\in N} u_k = H\left(G\right) + K \sum_{i\in G} w_i \ln \sum_{j\in N_i} u_j$$
$$H\left(G\right) = -K \sum_{i\in G} w_i \ln w_i$$
$$w_i \equiv \frac{u_{N_i}}{\sum_{k\in G} u_{N_k}}$$

QED.

Theil (1967, pp. 91, 290) has used Shannon's entropy based on income and sales as an indicator of income inequality and industry concentration, respectively. He does not show that property three is preserved.

The use of value for the construction of weights as in sections 2 and 3 is open to criticism. One could argue that types of goods for which the value traded is the same are all equally unique in their ability of generating output or utility. This line of argument ignores the possible existence of substitutes. Suppose the value traded of two types of goods, A and B, is identical but that there exists a good alternative only for good A. Now, by way of experiment, remove type A from the economy. The damage caused by this action will be limited as the substitute will take over the functions of A, albeit less efficiently. Removal of type B will have far greater consequences as it cannot be replaced by something else. Type B is more unique than type A even though they are being weighted equally.

The failure to explicitly take into account substitutability and uniqueness is not as disadvantageous as it might appear to be. Let us return to the 'fruit-basket' example given above and experiment by removing apples. Suppose red and green apples are very close substitutes. The removal of all green apples then would lead to a reduction in entropy of 0.231 as red apples, oranges and bananas all have a weight of 1/3. If there would be no substitute for green apples, the weights would be 1/5 for red apples and 2/5 each for oranges and bananas. The corresponding decline in entropy is 0.275 which is larger than 0.231, reflecting that the loss in variety is larger when there are no substitutes. Substitution effects do have an influence on value-weighted entropy, although this influence might not be as large as would be desirable.

Diversity value functions like those suggested by Weitzman (1992, 1998) and Nehring and Puppe (2002) explicitly take the similarity of objects as the foundation for the valuation of diversity. Their approach has a substantial advantage in that it allows for different degrees of substitutability between products. A potential disadvantage of these diversity value functions is that quantities of types of objects (e.g. products) are deemed to be irrelevant for the value of diversity. From an economic point of view, the scarcity of objects should matter for the value of diversity. What good is the existence of red apples if the worldwide annual production is just ten pieces?

5 Concluding remarks

The indices proposed in this paper exactly and unconditionally pass the product test of index number theory. Because the indices are measured over single sets of goods they can be used for the comparison of sets of goods that do not contain the same types of products. Changes in product variety do not lead to biases in the indices.

It has been shown that the indices can be used for the decomposition of productivity growth into scale and non-scale effects provided that there is a single output and that the production function is homothetic. In a similar fashion, the indices allow for the decomposition of changes in the cost of living into the effect of changes in the price index of consumer goods and the effect of changes in the 'efficiency' in consumption, conditional upon the utility function being homothetic.

In the special case of CES functions, the indices allow for the separation of the effects of product variety and product quality on productivity and the cost of living. For this reason, the proposed indices can be considered an improvement over the indices proposed by Feenstra, Markusen, and Zeile (1992) and Feenstra (1994).

Shannon's (1948) entropy possesses properties that make it suitable as an index of product variety. The apparent incompatibility with the diversity value functions of Weitzman (1992, 1998) and Nehring and Puppe (2002) is a subject for further research.

A Multiplicative decomposition of additive indices

Theorem 3 *Any index u measured over a set N that satisfies*

$$u_N = \sum_{i \in N} u_i$$
$$u_i = \prod_{j \in M} x_{ij}^{a_j}$$

can be decomposed into indices X_j for all $j \in M$ such that

$$u_N = \exp[H] \prod_{j \in M} X_j^{a_j}$$

where

$$H = -\sum_{i \in N} w_i \ln w_i$$
$$X_j = \prod_{i \in N} x_{ij}^{w_i}$$
$$w_i \equiv \frac{u_i}{u_N}.$$

Proof.

$$\begin{aligned}
\ln\left(u_N\right) &= H + \sum_{j \in M} \ln\left(X_j^{a_j}\right) = \sum_{i \in N} w_i \ln \frac{1}{w_i} + \sum_{i \in N} w_i \ln\left[\prod_{j \in M} x_{ij}^{a_j}\right] \\
&= \sum_{i \in N} w_i \ln\left[\frac{1}{w_i} \prod_{j \in M} x_{ij}^{a_j}\right] \\
&= \sum_{i \in N} w_i \ln\left[\frac{\prod_{j \in M} x_{ij}^{a_j}}{\prod_{j \in M} x_{ij}^{a_j} \Big/ \sum_{i \in N} \prod_{j \in M} x_{ij}^{a_j}}\right] \\
&= \ln\left[\sum_{i \in N} \prod_{j \in M} x_{ij}^{a_j}\right] \sum_{i \in N} w_i = \ln\left[\sum_{i \in N} u_i\right]
\end{aligned}$$

QED.

B Simulation results

The framework on which the simulations are based, starts with utility optimizing consumers that have preferences given by the CES function $C = \left(\sum_i y_i^{\alpha}\right)^{\frac{1}{\alpha}}$. Each good is manufactured by only one firm with the use of labor, $y_i = b_i l_i$, where the 'technical efficiency' of firms is heterogenous. In this setting of monopolistic competition, prices are markups over the wage rate, $p_i = w / \left(\alpha b_i\right)$. The consumption index (C) is the numeraire.

Two sets of simulations have been performed. In the first set the number of firms increases by one each period; in the second set the number of firms decreases by one each period. The b_i are distributed lognormally across firms with mean 1 and variance 0.25 but are constant over time. The number of firms, n, starts from 1000, the total labor force $L = \sum_i l_i$ is 1000, and α is set to 0.75. Each of the two simulation setups is replicated 500 times. The results in levels for quantity, price, and variety indices are displayed in figures

2, 3, and 4, respectively. The thick lines correspond to the median of the index, the thin lines correspond to the 5% and 95% quantiles implying that 90% of the replications lie between the grey lines.

Besides Q, the following other quantity indices are reported in figure 2: Laspeyres (QL), Paassche (QP), Fisher (QF) and the ordinary sum of quantities, $Y = \sum_i y_i$. The Paassche quantity index uses current prices as weights and is therefore not sensitive to increases in the set of products as no arbitrary decisions have to be made about the prices of new products in the previous period. In contrast, the Laspeyres quantity index does rely on prices from the previous period and is therefore biased if the number of goods is increasing. The Fisher index being a geometric average of an unbiased and a biased index is also biased but less so than the Laspeyres index. This is reflected in the first part of figure 2. Q and QP, the two unbiased indices in this case, remain very close to each other, while QL and QF diverge.

The bottom part of figure 2 applies to the case where the number of products is decreasing. For contracting product sets, the results are the reverse of those for expanding product sets. With a decreasing number of products, the Paassche index is biased whereas the Laspeyres index is unbiased as the latter relies on previous prices rather than current prices. Now the pair of unbiased indices is Q and QL. The conclusion to be drawn from figure 2 is that Q is the only index that is unbiased for both expanding and contracting product sets and that the Fisher index is biased in both cases.

Figure 3 shows that the results for P and the Paassche, Laspeyres, and Fisher price indices are equivalent to the results for Q and the other quantity indices. P is the only price index that is unbiased for both expanding and contracting product sets.

Figure 4 plots both N and the level equivalent of the Feenstra variety index (NFe), which is defined as

$$\Delta_{0,1} \ln (\text{NFe}) = \ln \left(\frac{\sum_{i \in G^0} p_i^0 x_i^0 / \sum_{i \in (G^0 \cap G^1)} p_i^0 x_i^0}{\sum_{i \in G^1} p_i^1 x_i^1 / \sum_{i \in (G^0 \cap G^1)} p_i^1 x_i^1} \right),$$

(see Feenstra et al (1999)). The indices are approximately equal as is to be expected in a CES context with constant b_i. In order to see whether this approximate equality holds when the b_i are changing over time, a third and fourth set of simulations have been done with the b_i following random walks of the form $b_{t,i} = z_{t,i} b_{t-1,i}$ with $z \sim LN(1, 0.25)$. The results are displayed in figure 5.

The presence of random walks has a reducing effect on N but not on the Feenstra index, which stays close to the number of products. The mechanism underlying this result is that random walks cause the differences between firms to increase over time, inducing concentration of production. This concentration effect is picked up by N but not by NFe as the former takes into account changes in the value of one *existing* product vis-à-vis another *existing* product whereas the latter does not.

The MATLAB code used for these simulations can be downloaded from http://meritbbs.unimaas.nl/straathof/.

C Application to CRESH functions

The homothetic constant ratios of elasticity of substitution (CRESH) production function devised by Hanoch (1971) differs from the general homothetic functions of section 3 in that its output is an implicit function of the inputs. Below it is shown that for the CRESH function results can be found similar to those of section 3.

The CRESH function is defined as

$$G(y,\mathbf{x}) = \sum_{i=1}^{n} \theta_i \left[x_i g(y)\right]^{\delta_i} - 1 \equiv 0.$$

The derivative of y with respect to x_j can be found using the implicit function theorem.

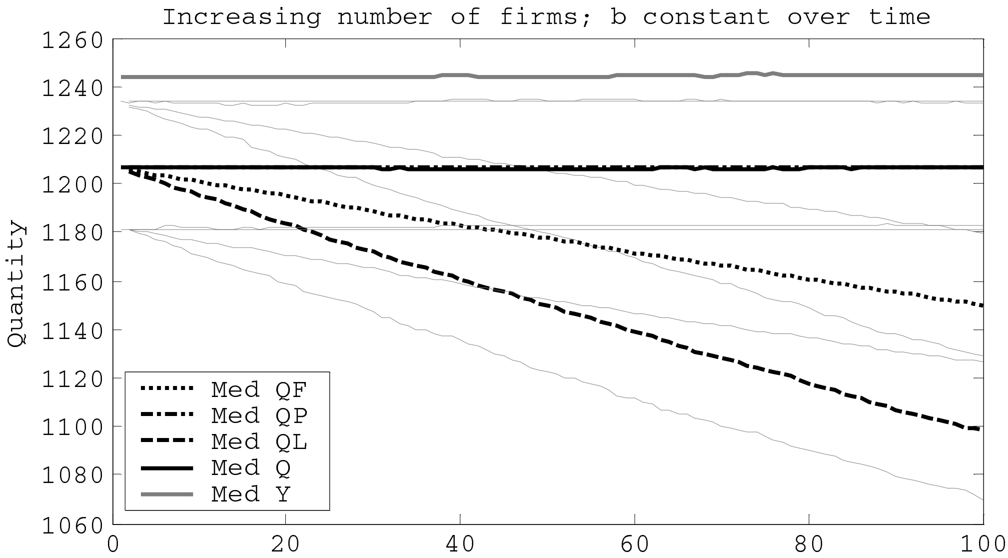
$$\frac{\partial y}{\partial x_j} = -\frac{G_{x_j}}{G_y} = \frac{\theta_j \delta_j {x_j}^{\delta_j - 1} g(y)^{\delta_j}}{\sum_{i=1}^{n} \theta_i \delta_i {x_i}^{\delta_i} g(y)^{\delta_i - 1} g'}$$

The first order condition for profit maximization ensures that $p_j = \left(1+\varepsilon_{p_y y}\right) \frac{\partial y}{\partial x_i} p_y$. This leads to the following expression for expenditure shares:

$$\begin{aligned} w_i &= \frac{\frac{\theta_i \delta_i {x_i}^{\delta_i - 1} g(y)^{\delta_i}}{\sum_i \theta_j \delta_j {x_j}^{\delta_j} g(y)^{\delta_j - 1} g'} x_i}{\sum_j \frac{\theta_j \delta_j {x_j}^{\delta_j - 1} g(y)^{\delta_j}}{\sum_k \theta_k \delta_k {x_k}^{\delta_k} g(y)^{\delta_k - 1} g'} x_j} \\ &= \frac{\theta_i \delta_i {x_i}^{\delta_i} g(y)^{\delta_i}}{\sum_j \theta_j \delta_j {x_j}^{\delta_j} g(y)^{\delta_j}}. \end{aligned}$$

We use these expenditure shares for the construction of an index of the derivatives $\partial y / \partial x_j$.

$$\begin{aligned} F' &\equiv \prod_j \left(\frac{g(y)}{g' x_j} \frac{\theta_j \delta_j {x_j}^{\delta_j} g(y)^{\delta_j}}{\sum_{i=1}^{n} \theta_i \delta_i {x_i}^{\delta_i} g(y)^{\delta_i}} \right)^{w_j} \\ &= \frac{g(y)}{g' X \sum_i \theta_i \delta_i {x_i}^{\delta_i} g(y)^{\delta_i}} \prod_j \left(\theta_j \delta_j {x_j}^{\delta_j} g(y)^{\delta_j} \right)^{w_j} \end{aligned}$$



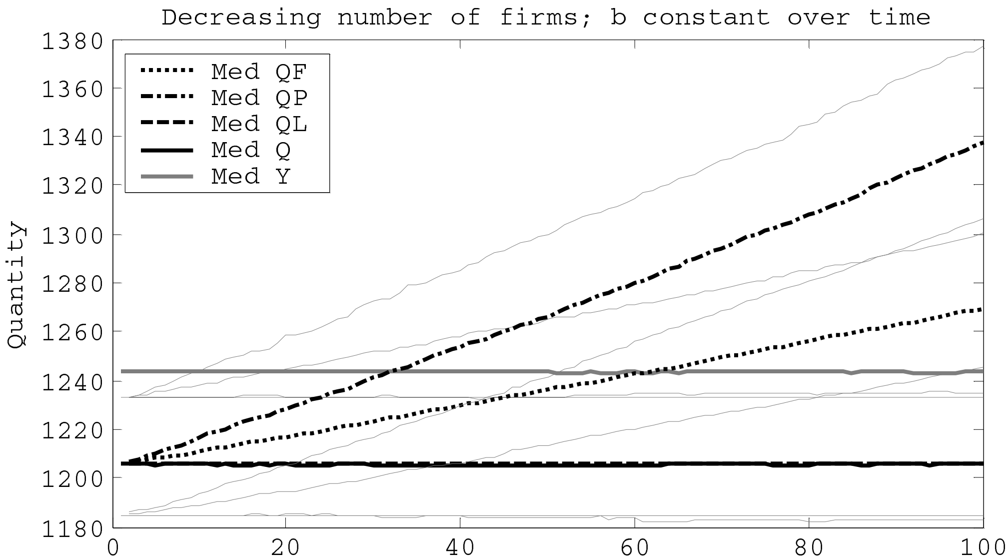
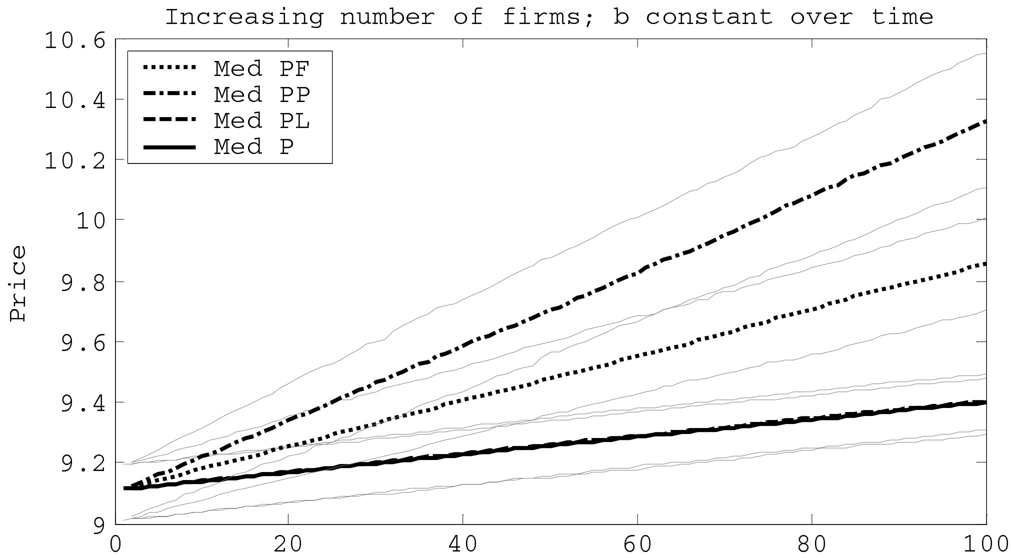


Figure 2: Simulation results for quantity indices



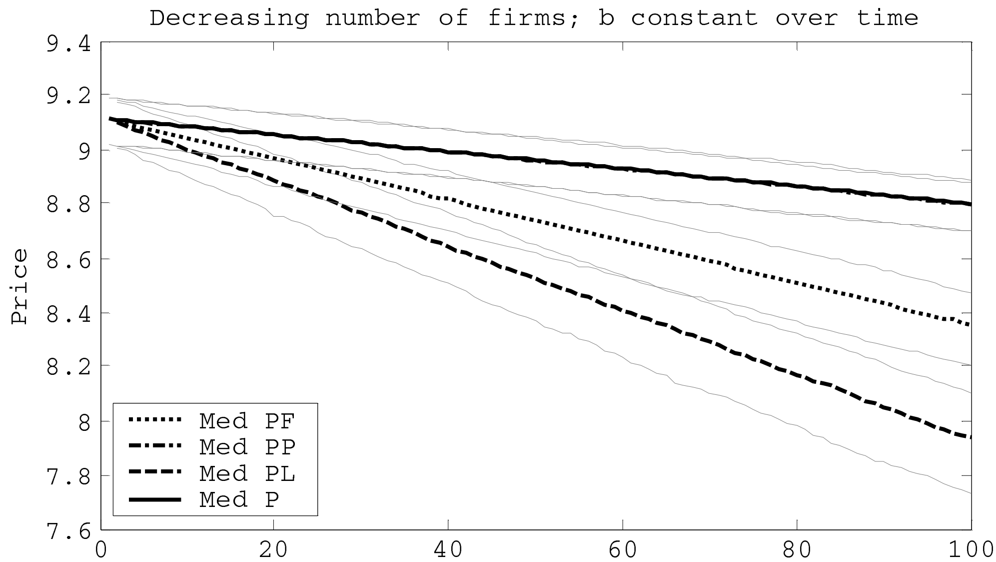


Figure 3: Simulation results for price indices

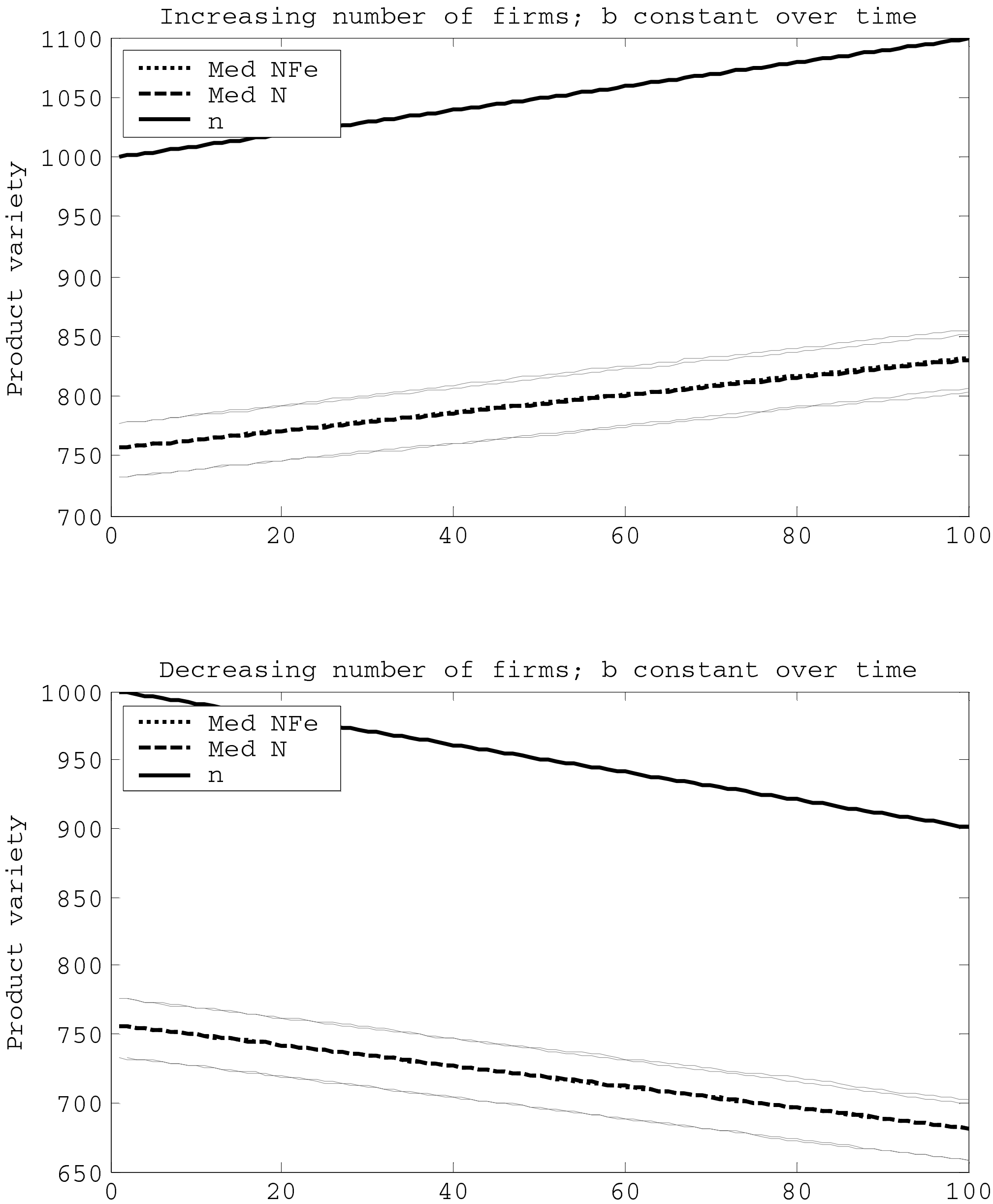


Figure 4: Simulation results for variety indices

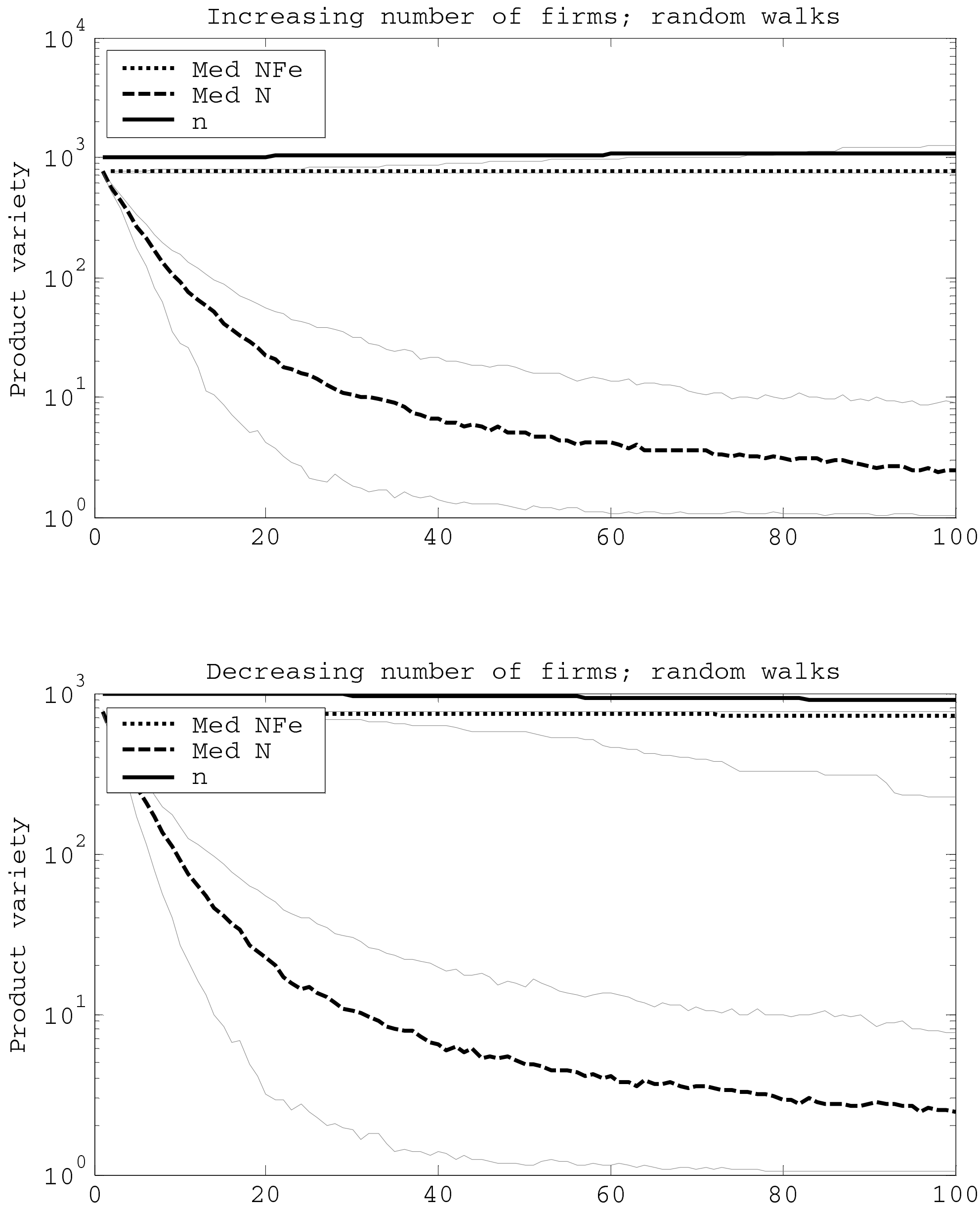


Figure 5: Variety indices: random walks

The summation below the line can be decomposed using the same weights as above.

$$\sum_i \theta_i \delta_i {x_i}^{\delta_i} g\left(y\right)^{\delta_i} = N \prod_i \left(\theta_i \delta_i {x_i}^{\delta_i} g\left(y\right)^{\delta_i}\right)^{w_i}$$

Insert this expression back into F' to get an index for productivity that is of identical form as that in subsection 3.1.

$$F' = \frac{g\left(y\right)}{g' X N \prod_i \left(\theta_i \delta_i {x_i}^{\delta_i} g\left(y\right)^{\delta_i}\right)^{w_i}} \prod_j \left(\theta_j \delta_j {x_j}^{\delta_j} g\left(y\right)^{\delta_j}\right)^{w_j}$$

$$\frac{y}{Q} = \varepsilon_{gy}^{-1} F'$$

The COLI corresponding to the CRESH function is

$$c = E/y = \frac{NPX}{\varepsilon_{gy}^{-1} F' Q} = \varepsilon_{gy} P/F'.$$

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