

AN EVOLUTIONARY MODEL WITH MYOPIC LEARNING

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Draft version, comments appreciated

June, 1994

MERIT Research Memorandum 2/94-019

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Abstract

We examine a dynamical system consisting of two distinct, but interactive, subsystems, namely population dynamics and learning dynamics. The population dynamics formalize that the population shares of fitter groups increase relatively to those of less fit groups. The learning dynamics describe that each subgroup adapts its strategy, by placing more weight on activities contributing more than average to its fitness, meanwhile decreasing weights on activities contributing less than average.

A saturated equilibrium is a dynamic equilibrium where no subgroup has above-average fitness, and all subgroups employ best-reply strategies to the population share weighted average strategy. We demonstrate that if a trajectory converges from the interior of the state space, then its limit point is a saturated equilibrium. An evolutionary stable equilibrium is a saturated equilibrium attracting all trajectories starting in a certain neighborhood of it. The properties of the saturated equilibrium and the evolutionary stable equilibrium suggest that these concepts are adequate dynamic generalizations of the Nash-equilibrium and the evolutionary stable strategy of the standard models.



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² Participants in a game theory meeting at the University of Limburg are thanked for comments.

1. Introduction

Economics has embraced game theory as an important tool for the analysis of strategic interaction. The main concept forwarded by noncooperative game theory as a predicted outcome under the assumption of perfect rationality of the agents, is the Nash-equilibrium. There are however, some difficulties with respect to using game theory in economics. Firstly, the rationality requirements of game theory are rarely met in an economical context. Secondly, there may exist multiple Nash-equilibria and it is not always clear how to select among them. The acknowledgement in game theory of these difficulties has led to a reexamination of rationality requirements, as well as to a vast array of literature dealing with less stylized settings. Furthermore, various refinements of the Nash-equilibrium concept have been formulated, each with its own merit. Evolutionary game theory has gained considerable popularity, since not only are the Nash-equilibria specified, but also the dynamic process which may lead to these outcomes, is defined. The rationality requirements under which evolutionary dynamics may converge to a Nash-equilibrium, are generally remarkably low.

In evolutionary game theory, subgroups within a population traditionally play genetically predetermined strategies. As a result, the population dynamics depend solely on the composition of the population [cf. Van Damme (1991)]. The population dynamics favor the fitter subgroups over the less fit, implying that the population shares of the fitter (less fit) subgroups increase (decrease). Thus, the strategies used by the more (less) successful subgroups gain more (less) weight in the population share weighted average strategy. Hence, changes in average behavior may be observed on an aggregate level, which may appear as if the population were learning. The replicator dynamics, which are widely used as population dynamics, converge only to Nash-equilibria for standard models in evolutionary game theory. Furthermore, an evolutionary stable strategy, one of the strongest refinements of the Nash-equilibrium concept, is an attractor¹ for the replicator dynamics [cf. Taylor and Jonker (1978)]. These properties inspired interpretations of these dynamics as an inductive learning process. In reality though, none of the subgroups in the standard models in evolutionary game

¹ An attractor is a dynamic equilibrium to which all trajectories reaching a neighborhood of this equilibrium, converge.

theory, alter their strategies, nor are they able to do so. Therefore, we can only speak about learning in an indirect sense. This should not come as a surprise, since the original evolutionary models were indeed intended to describe the evolution of populations of individuals whose behavior is primarily genetically predetermined. This precludes the possibility of direct behavior modification of individuals is impossible. We regard the latter as a rather unsatisfactory aspect when modelling the strategic behavior of more sophisticated beings.

We assume that each subgroup in a population, is able to learn in a manner which is quite similar to a process proposed by Friedman (1991). The subgroups monitor the success and failure of their behavioral strategy in its strategic environment, and alter their strategies gradually by placing more (less) weight on those strategies which yield a higher (lower) payoff. We interpret this adaptive process on the strategies as an inductive learning process. Meanwhile, the composition of the population changes under evolutionary influences, implying that the fitter subgroups grow in population share relative to the less fit ones. We formulate a dynamic system consisting of population and learning dynamics for all subgroups in the population. The dynamics of the subsystems are quite similar, however we emphasize the conceptual differences between these interactive subsystems.

We apply generalizations of the saturated equilibrium and the evolutionary stable equilibrium developed in Joosten (1993), to the present model. A saturated equilibrium is a rest point of the dynamical system, where all subgroups with positive population share have equal payoff. All groups with below-average payoff have population share zero and no group with above-average payoff exists. All subgroups present at any point in time, employ only strategies that are myopic best-reply strategies to the population share weighted average strategy. We show that an evolutionary stable equilibrium is an attractor of the dynamics for some subset of the state space. Whenever the system reaches a point of this subset it converges to this evolutionary stable equilibrium in a 'well-behaved' manner. A limit point is an attractor for at least one trajectory starting in a nonequilibrium point. We show that every limit point for a trajectory starting in the interior of the state space, is a saturated equilibrium.

In Section 2 we introduce the model, and in Section 3 we relate our model to literature on population and learning dynamics. In Section 4 we prove existence of certain types of equilibria, and show connections between different types of rest points of the

dynamics. In Section 5 we expand somewhat on the connections, as well as on the discrepancies, between strategic stability and dynamic stability in our model. Section 6 briefly mentions some extensions and Section 7 concludes.

2. Combining population dynamics and learning dynamics

The model which we present in this section, may be regarded as a synthesis between the model of Friedman (1991) and an earlier model of ours, namely Joosten (1993). Friedman (1991) analyses classes of dynamics on the Cartesian product of the strategy spaces of a finite number of agents, and the equilibria of these dynamics. In Joosten (1993), population dynamics and their dynamic equilibria are examined for very general evolutionary settings. The approach followed in the latter paper, is generalized in the present one to incorporate Friedman-type learning dynamics.

We consider a population with $n+1$ distinguishable, interacting subgroups, where $n \in \mathbb{N}$. Each subgroup plays a (possibly) mixed behavioral strategy. The population and its subgroups are assumed to consist of very large numbers of randomly mixing individuals. Let $I^{n+1} = \{1, 2, \dots, n+1\}$ and let $S^p = \{y \in \mathbb{R}^{p+1} \mid \sum_i y_i = 1, y_i \geq 0 \text{ for all } i \in I^{p+1}\}$. The state space, the Cartesian product of one n -dimensional unit simplex and $n+1$ m -dimensional unit simplices, is denoted by $\Xi = S^n \times \prod_i S^m$. A generic element of Ξ is denoted by an ordered pair (x, s) , where x denotes an $(n+1)$ -dimensional vector of population shares, and s denotes the Cartesian product of $n+1$ $(m+1)$ -dimensional vectors, representing the strategies employed by the subgroups. Let A^1, A^2, \dots, A^{n+1} be a collection of $n+1$ fixed $(m+1) \times (m+1)$ -matrices. The population strategy at $(x, s) \in \Xi$ is given by $S(x, s) = \sum_j x_j s^j$. We define the payoff function $\pi: \Xi \rightarrow \mathbb{R}^{n+1}$, as follows

$$\pi^i(x, s) = s^i A^i S(x, s) \quad \text{for all } i \in I^{n+1}. \quad (\mathbf{1})$$

The payoff function attributes at every state to each subgroup a payoff, depending on the strategy employed by the subgroup, on the strategies used by other subgroups, and on the composition of the population. The payoff to a subgroup can be interpreted as its fitness, i.e.

a measure of potential of this subgroup to reproduce or grow.

The composition of the population evolves in time. The population share of each subgroup changes in accordance with the fitness of that subgroup relative to the average fitness of the population. The relative fitness function $f_i: \Xi \rightarrow \mathbb{R}^{n+1}$ for every $i \in I^{n+1}$, is given by

$$f_i(x,s) = \pi^i(x,s) - \sum_j x_j \pi^j(x,s) \quad \text{at each } (x,s) \in \Xi. \quad (2)$$

Before introducing the learning dynamics, we intend to compare for each subgroup, the contribution of each pure action to the payoff to the subgroup. Let therefore, for all $i \in I^{n+1}$, $h \in I^{m+1}$, $\pi_h^i(x,s) = e_h^T A^i S(x,s)$, where e_h is the h -th unit vector in \mathbb{R}^{m+1} , denote the payoff on pure action h at each state $(x,s) \in \Xi$. Let for each $i \in I^{n+1}$ the function $g^i: \Xi \rightarrow \mathbb{R}^{m+1}$ be defined by

$$g_h^i(x,s) = \pi_h^i(x,s) - \sum_k s_k \pi_k^i(x,s) \quad \text{for all } h \in I^{m+1}. \quad (3)$$

Note that $\pi^i(x,s) = \sum_k s_k \pi_k^i(x,s)$. We call such a function $g^i: \Xi \rightarrow \mathbb{R}^{m+1}$ defined by (3) a marginal payoff function. It can be shown that $x^T f(x,s) = 0$ and $(s^i)^T g^i(x,s) = 0$ for all $i \in I^{n+1}$ and all $(x,s) \in \Xi$. In general, there exist groups with above and below average fitness. Similarly, there exist pure strategies yielding above- as well as below-average payoffs in general.

We call $G^i: \Xi \rightarrow \mathbb{R}^{m+1}$ weakly compatible for $g^i: \Xi \rightarrow \mathbb{R}^{m+1}$ if

- a) G^i is continuous for all $(x,s) \in \Xi$,
- b) $\sum_h G_h^i(x,s) = 0$ for all $(x,s) \in \Xi$,
- c) $G_h^i(x,s) = 0$ for all $h \in I^{m+1}$, whenever $x_h = 0$,
- d) $G_h^i(x,s) \geq 0$ whenever $s_h^i = 0$,
- e) $\text{sgn } G_h^i(x,s) = \text{sgn } g_h^i(x,s)$ otherwise.

For all $y \in \mathbb{R}$: $\text{sgn } y = +1$ if $y > 0$, $\text{sgn } y = 0$ iff $y = 0$, and $\text{sgn } y = -1$ iff $y < 0$. The learning process is to be regarded as an inductive rather than a deductive process, hence only subgroups that are present in the population, are able to learn from the strategic environment, which is formalized in Restriction (c).

By now, we are in the position to describe the dynamical system. The population dynamics are represented by the following $(n+1)$ -system of (autonomous) differential equations:

$$dx/dt = H(x,s), \quad \text{for all } (x,s) \in \Xi, \quad (4)$$

where $dx/dt = (dx_1/dt, dx_2/dt, \dots, dx_{n+1}/dt)$ and $H(x,s) = (H_1(x,s), H_2(x,s), \dots, H_{n+1}(x,s)) = (x_1 f_1(x,s), x_2 f_2(x,s), \dots, x_{n+1} f_{n+1}(x,s))$. These population dynamics are known in the literature under the name of replicator dynamics. The learning dynamics are described by the following $(n+1) \times (m+1)$ -system of (autonomous) differential equations at each $(x,s) \in \Xi$:

$$ds^i/dt = G^i(x,s) \quad \text{for all } i \in I^{n+1}, \quad (5)$$

where $ds^i/dt = (ds^i_1/dt, ds^i_2/dt, \dots, ds^i_{m+1}/dt)$, and for all $i \in I^{n+1}$ it holds that $G^i: \Xi \rightarrow \mathbb{R}^{m+1}$ is weakly compatible for the function g^i .

A trajectory $\{(x(t), s(t))\}_{t \geq 0} \subset \Xi$, is a sequence of states generated by the dynamical system described by Eqs. (4) and (5). The dynamical system is length preserving, which implies for each trajectory $\{(x(t), s(t))\}_{t \geq 0}$ that $\sum_i x_i(t) = 1$ and $\sum_h s^i_h(t) = 1$ for all $i \in I^{n+1}$, for all $t \geq 0$ provided $((x(0), s(0)) \in \Xi$. This follows from the definitions of the replicator dynamics and the weakly compatible learning dynamics, since for all $(x,s) \in \Xi$

$$\sum_i H_i(x,s) = 0, \text{ and } \sum_h G^i_h(x,s) = 0 \text{ for all } i \in I^{n+1}.$$

Every trajectory starting in Ξ remains in Ξ , since for all $i \in I^{n+1}$, $h \in I^{m+1}$:

$$\begin{aligned} x_i = 0 & \quad \text{implies } dx_i/dt = 0, \\ s^i_h = 0 & \quad \text{implies } ds^i_h/dt \geq 0. \end{aligned}$$

Hence, no trajectory traverses any boundary of the state space Ξ from Ξ .

3. Learning dynamics and population dynamics: an overview of and connections with the literature

The population dynamics of Eq. (4) were introduced in game theory by Taylor and Jonker (1978) under the name of replicator dynamics. The assumption behind Eq. (1), is that the probability for an individual to be matched with an individual from a certain subgroup is equal to the population share of this subgroup. Hence, the expected strategy against which each individual in each subgroup plays is the population strategy. Our model has a position somewhere in between the asymmetric games in the terminology of Hofbauer and Sigmund (1988), and the so-called viscous population models [e.g. Myerson *et al.* (1991)]. In the former no direct strategic interaction of any subgroup with itself occurs. In the latter each individual has a higher tendency to interact with members of its own subgroup. We justify the use of deterministic differential equations in (4) and (5) as an approximation of the stochastic dynamics by our large-numbers-assumption.

The individuals in our model are not born with a strategy 'preprogrammed' by nature. Instead, they are born with a capacity to learn by induction or imitation, a capacity to figure out how to improve (*ceteris paribus*) in any situation that might occur. On an aggregate level, this results in a dynamic process for each subgroup, which we interpret as a learning process. The learning process is closely related to the dynamics of Friedman (1991). Neither the composition of the population, nor any of the strategies used by the subgroups, remain constant in general, hence the population strategy may change in a complicated manner. The subgroups adapt their strategies 'myopically' [Kalai and Lehrer (1993)] based on their current payoffs, as if the strategic environment were not changing. Instead of myopic, we also found the term 'naive' in the literature for similar ideas on adaptive learning processes [e.g. Eichberger *et al.* (1993)]. Since the learning process does not forecast and it has no recording of distant events, it is not sophisticated [cf. Milgrom and Roberts (1991)].

The learning process can be seen as rational within the perceptual and computational limitations of the agents. Simon (1956) distinguishes subjective and objective rationality. Subjectively rational behavior is behavior which is rational, given the perceptual and evaluational premises of the subject. Objectively rational behavior is behavior which is rational in the 'usual' sense. The inductive learning process modeled in this paper, is not

objectively rational, but it can be regarded as subjectively rational. The subgroups favor namely the strategies of which they expect to improve their payoffs based on the information which they possess and which consists of their current payoffs.

We provide informal motivations for the learning process, and we regard these motivations as complementary. The first motivation is inspired by Selten's (1991) remarks on rote-learning. Each individual in a subgroup has a certain strategy and receives a certain payoff. If this activity yields a high (low) payoff, then the probability to discontinue this activity is low (high). Assuming that all other activities have equal probabilities to be adopted, this could generate an effect as if the subgroup were learning [see also Friedman and Rosenthal (1986)]. Other terms connected with behavior modification similar to rote-learning, are conditioning and reinforcement, many aspects of which have been and are studied in psychology. The second motivation, influenced by Selten's (1991) remarks on learning by imitation, is that individuals imitate the strategies of other individuals in the same subgroup. If more successful individuals have a higher probability of being imitated than less successful ones, this could also generate an effect as if the population were learning consciously. The final motivation is that individuals receiving low payoffs may switch to arbitrary alternatives, not knowing the payoffs on these alternatives. The latter could be called 'learning by experimentation'.

Many processes are advanced in the literature to describe different types of learning. Quite a few employ best-response dynamics [e.g. Gilboa and Matsui (1991)]. Under best-response dynamics only the weights on the pure strategies that yield the highest payoffs in the current strategic environment, are increased. The weights on all other pure strategies are decreased by the same proportion. Hence, it may very well happen that the weight on a 'nearly optimal' pure strategy is decreased by the same proportion as the weight on the 'worst' pure strategy. Our learning dynamics differ from these best-response dynamics. The weights on all pure strategies that give above-average payoffs, are increased, whereas the weights on all strategies yielding below-average payoffs, are simultaneously decreased.

4. Dynamic equilibria and limit points

In this section, we examine equilibria of the dynamical system described in Section 2. The state $(y,u) \in \Xi$, is a dynamic equilibrium if $H_i(y,u) = 0$ and $G_h^i(y,u) = 0$ for all $i \in I^{n+1}$, $h \in I^{m+1}$. Occasionally, we call a dynamic equilibrium a rest point. Let $p,q \in S^m$ and let $i \in I^{n+1}$, then p is a myopic best reply for subgroup i against q if $p_h > 0$ implies $e_h A^i q = \max_j e_j A^i q$. In that case we will write $p \in MBR_i(q)$, whereas if $u^i \in MBR_i(q)$ for all $i \in I^{n+1}$, we will use the notation $u \in MBR(q)$. *If a subgroup with population share equal to one, employs a strategy which is a myopic best reply against itself, then the dynamical system is in equilibrium.*² It should be noted that this holds for arbitrary strategies used by the other subgroups, hence there may exist a multitude of these equilibria.

A dynamic equilibrium $(y,u) \in \Xi$ is stable, if for any (open) neighborhood $U \subset \Xi$ of (y,u) , there exists $V \subset U$ such that any trajectory starting in V remains in U . A stable equilibrium is called asymptotically stable, if additionally a neighborhood $W \subseteq V$ of (y,u) exists such that any trajectory starting in W converges towards (y,u) [cf. Hirsch and Smale (1974)]. We have shown that there exist many dynamic equilibria. It need not be that any of these equilibria is stable, nor that any trajectory converge to any such equilibrium. A state (y,u) is called a limit point if there exists a trajectory $\{(x(t),s(t))\}_{t \geq 0} \subset \Xi$, satisfying $(x(0),s(0)) \neq (y,u)$ and $(y,u) = \text{Lim}_{t \rightarrow \infty} (x(t),s(t))$. It need not be that a trajectory starting in a point which is not an equilibrium, converges, nor need it be that any such trajectory converges. As a consequence, limit points may not exist.

A state $(y,u) \in \Xi$ is a saturated equilibrium if for all $i \in I^{n+1}$, $h \in I^{m+1}$, it holds that

$$f_i(y,u) \leq 0 \text{ and } g_h^i(y,u) \leq 0. \quad (6)$$

We firstly prove that each saturated equilibrium is a rest point for the dynamic system described by Eqs. (4) and (5), then we demonstrate that there always exists at least one saturated equilibrium. Let $C(z) = \{j \in I^{p+1} \mid z_j > 0\}$ for every $z \in \mathbb{R}^{p+1}$, $p = m,n$.

² The proof is omitted.

Proposition 4.1. *Each saturated equilibrium is a dynamic equilibrium.*

Proof. Let $(y,u) \in \Xi$ be a saturated equilibrium. Let $i \in I^{n+1}$, since $f_j(y,u) \leq 0$ for all $j \in I^{n+1}$, it holds that $0 = \sum_j y_j f_j(y,u) \leq y_i f_i(y,u) \leq 0$. Since $i \in I^{n+1}$ was taken arbitrarily, it follows that $H_j(y,u) = 0$, for all $j \in I^{n+1}$.

Note that $y_i = 0$ implies $G_k^i(y,u) = 0$ for all $k \in I^{m+1}$, suppose therefore $i \in C(y)$, and $k^* \in C(u^i)$ exists such that $G_{k^*}^i(y,u) < 0$.

Then l^* exists, satisfying $G_{l^*}^i(y,u) > 0$, implying $g_{l^*}^i(y,u) > 0$.

This leads to a contradiction, hence $G_k^i(x,s) = 0$ for all $k \in I^{m+1}$.

Since $i \in C(y)$ was taken arbitrarily, this completes the proof. \square

Proposition 4.2. *For an arbitrary relative fitness function, and for arbitrary relative marginal payoff functions, there exists a saturated equilibrium.*

Proof. Let F_0 be the point-to-set-mapping from Ξ to the subsets of S^n , which for every $(x,s) \in \Xi$ is defined by

$$F_0(x,s) = \text{conv}\{e(j) \in \mathbb{R}^{n+1} \mid \pi^j(x,s) = \max_i \pi^i(x,s)\}.$$

Let for each $i \in I^{n+1}$, F_i be the point-to-set-mapping from Ξ to the subsets of S^m , which for every $(x,s) \in \Xi$ is defined by

$$F_i(x,s) = \text{conv}\{e(j) \in \mathbb{R}^{m+1} \mid \pi_j^i(x,s) = \max_h \pi_h^i(x,s)\}.$$

Let F be the point-to-set-mapping from Ξ to the subsets of Ξ , which for every $(x,s) \in \Xi$ is defined by $F(x,s) = \prod_{i=0,1,\dots,n+1} F_i(x,s)$. The mappings F_0, F_1, \dots, F_{n+1} are upper-semicontinuous by construction, implying that F is upper-semicontinuous as well.

Since the state space is compact and convex, Kakutani's (1941) theorem applies.

Hence, $(y,u) \in \Xi$ exists, satisfying $(y,u) \in F(y,u)$.

Let $(y,u) \in F(y,u)$, then $j \in C(y)$ implies $\pi^j(y,u) = \max_i \pi^i(y,u)$.

Let $c_0 = \max_i \pi^i(x,s)$, then

$$\begin{aligned} f_k(y,u) &= \pi^k(y,u) - \sum_i y_i \pi^i(y,u) = \\ &= \pi^k(y,u) - \sum_{j \in C(y)} y_j c_0 = \\ &= \pi^k(y,u) - c_0 \leq 0, \text{ for all } k \in I^{n+1}. \end{aligned}$$

Take $k \in I^{n+1}$, then $h \in C(u^k)$ implies $\pi_h^k(y,u) = \max_j \pi_j^k(y,u)$.

Let $\max_j \pi_j^k(y,u) = c_k$, then (writing $C(u,k)$ for $C(u^k)$)

$$\begin{aligned} g_h^k(y,u) &= \pi_h^k(y,u) - \sum_j u_j^k \pi_j^k(y,u) = \\ &= \pi_h^k(y,u) - \sum_{j \in C(u,k)} u_j^k c_k = \\ &= \pi_h^k(y,u) - c_k \leq 0 \text{ for all } h \in I^{m+1}. \end{aligned} \quad \square$$

Nachbar (1991) has shown in a different setting that for a large class of 'evolutionary' dynamics any limit point for a trajectory from the interior of the strategy space, is a Nash-equilibrium. From the continuity of the right hand sides of Eqs. (4) and (5), combined with the compactness of Ξ , it follows directly that any limit point (y,u) is a dynamic equilibrium and that $(y,u) \in \Xi$. In the following proposition, we list further properties of a limit point.

Proposition 4.3. *Let (y,u) be the limit point for some trajectory $\{(x(t),s(t))\}_{t \geq 0} \subset \Xi$, then the following properties hold:*

$$i \in C(y) \text{ implies } \quad h \in C(u^i): \quad \pi_h^i(y,u) = \pi^i(y,u) = \sum_j y_j \pi_j^i(y,u) \quad (7)$$

$$h \in C(u^i): \quad \pi_h^i(y,u) \leq \pi^i(y,u) \quad (8)$$

$$i \in C(x(0)) \setminus C(y) \text{ implies } \quad h \in C(u^i): \quad \pi_h^i(y,u) = \pi^i(y,u) \leq \sum_j y_j \pi_j^i(y,u) \quad (9)$$

$$h \in C(u^i): \quad \pi_h^i(y,u) \leq \pi^i(y,u) \quad (10)$$

$$i \notin C(x(0)) \text{ implies } \quad y_i = 0 \text{ and } u^i = s^i(0) \quad (11)$$

Proof. Eq. (7) follows from the continuity of the functions H and G^i , for all $i \in I^{n+1}$.

Eq.(11) follows from Restriction (c). Eqs. (8) and (10) can be validated as follows.

Suppose for a given $i \in C(x(0))$, $h \in C(s(0)) \setminus C(u^i)$ exists such that $\pi_h^i(y,u) > \pi^i(y,u)$. Then an $\epsilon > 0$ and a set $B((y,u),\epsilon) = \{(x,s) \in \Xi \mid d_2((y,u),(x,s)) \leq \epsilon\}$

exist, such that $(x,s) \in B((y,u),\epsilon)$ implies $g_h^i(x,s) > 0$. Since $(y,u) =$

$\lim_{t \rightarrow \infty} (x(t),s(t))$ there exists $t^* \geq 0$ satisfying $d_2((y,s),(x(t^*),s(t^*))) < \epsilon$ and

$ds_h^i(t^*)/dt < 0$. The latter leads to a contradiction, since by the forward invariance of the system $s_h^i(t^*) > 0$, hence $(x(t^*),s(t^*)) \in B((y,u),\epsilon)$ implies $G_h^i(x(t^*),s(t^*))$

> 0 . This establishes (8) and (10).

Eq. (9) remains to be proven.

From the continuity of the functions G^i , combined with Restriction (e) of weakly compatible dynamics and the convergence, it follows that for all $h \in C(u^i)$, that $\pi_h^i(y,u) = \pi^i(y,u)$ for all $i \in C(x(0)) \setminus C(y)$.

Suppose however, $\pi^i(y,u) > \sum_j y_j \pi^j(y,u)$ for some $i \in C(x(0)) \setminus C(y)$, then an $\epsilon > 0$ and a set $B((y,u),\epsilon) = \{(x,s) \in \Xi \mid d_2((y,u),(x,s)) \leq \epsilon\}$, such that $(x,s) \in B((y,u),\epsilon)$ implies $f_i(x,s) > 0$. Since $\lim_{t \rightarrow \infty} (x(t),s(t)) = (y,u)$ there exists $t^* \geq 0$ satisfying $d_2((y,s),(x(t^*),s(t^*))) < \epsilon$ and $dx_i(t^*)/dt < 0$.

However, $\pi^i(x(t^*),s(t^*)) - \sum_j x_j(t^*) \pi^j(x(t^*),s(t^*)) > 0$ and $x_i(t^*) > 0$ by the forward invariance, imply $H_i(x(t^*),s(t^*)) > 0$. Since $(x(t^*),s(t^*)) \in B((y,u),\epsilon)$ we have shown a contradiction, which establishes (9). \square

Equations (7), (8), (9) and (10) imply that if a trajectory converges, then all subgroups present, employ only their myopic best-reply strategies against the population strategy in the long run. Not every subgroup generally survives as time goes to infinity, which occurs if the subgroup's myopic best-reply strategy against the population strategy yields a payoff which is permanently lower than the average payoff of the population. There is a connection between limit points and saturated equilibria. If, for a converging trajectory in Ξ , $C(x(0)) = I^{n+1}$, then its limit point is a saturated equilibrium. This leads to the following corollary.

Corollary 4.1. *Any trajectory starting in a nonequilibrium state where all subgroups have positive population share, converges only to a saturated equilibrium.*

Since an asymptotically stable equilibrium is a limit point for all trajectories starting within a certain neighborhood of it, it follows immediately that such an equilibrium is a saturated equilibrium. For the weaker concept of the stable equilibrium, the following is in order.

Proposition 4.4. *Every stable equilibrium is a saturated equilibrium.*

Proof. Let (y,u) be a stable equilibrium and suppose (y,u) is not a saturated equilibrium.

Then (y,u) is not in the interior of Ξ , since all interior equilibria are saturated.

Hence, either $i \in I^{n+1}$ exists satisfying $y_i = 0$ and $f_i(y,u) > 0$, or $j \in I^{n+1}$ and

$k \in I^{m+1}$ exist satisfying $u_k^j = 0$ and $g_k^j(y,u) > 0$.

Suppose $y_i = 0$ and $f_i(y,u) > 0$ for some $i \in I^{n+1}$. Then by the continuity of f , an $\epsilon > 0$ and a $\delta > 0$ exist such that $f_i(x,s) > \epsilon$ for all $(x,s) \in U$, where $U = \{(x,s) \in \Xi \mid \max_j |x_j - y_j| < \delta, j \in I^{n+1}\}$.

Let $\{(x(t),s(t))\}_{t \geq 0}$ be a trajectory with $(x(0),s(0)) \in U$, and $x_i(0) > 0$. Then $(x(t),s(t)) \in U$, implies $H_i(x(t),s(t)) > 0$ for all $t \geq 0$. Furthermore, there exists $k \geq 0$ such that $\max_j |x_j(k) - y_j| = \delta$. Hence, $\{(x(t),s(t))\}_{t \geq 0}$ leaves U . This contradicts (y,u) is stable.

Suppose $u_k^j = 0$ and $g_k^j(y,u) > 0$ for some $j \in I^{n+1}$, $k \in I^{m+1}$. Then by the continuity of g_k^j , an $\epsilon > 0$ and a $\delta > 0$ exist such that $g_k^j(x,s) > \epsilon$ for all $(x,s) \in U$, where $U = \{(x,s) \in \Xi \mid \max_l |s_l^j - y_l^j| < \delta, l \in I^{m+1}\}$.

Let $\{(x(t),s(t))\}_{t \geq 0}$ be a trajectory with $(x(0),s(0)) \in U$, and $x_j(0) > 0$ and $s_k^j(0) > 0$, then $(x(t),s(t)) \in U$, implies $G_k^j(x(t),s(t)) > 0$ for all $t \geq 0$.

Furthermore, there exists $k \geq 0$ such that $\max_j |x_j(k) - y_j| = \delta$.

Hence, $\{(x(t),s(t))\}_{t \geq 0}$ leaves U . This contradicts the stability of (y,u) . \square

Each equilibrium in the interior of Ξ is a saturated equilibrium, hence not every saturated equilibrium is stable. The links between stable equilibria and limit points is unclear in general. Stable equilibria need namely not be limit points, and limit points may be unstable.

An equilibrium $(y,u) \in \Xi$ is an evolutionary stable equilibrium if and only if a neighborhood $U \subset \Xi$ of (y,u) exists, satisfying for all $(x,s) \in U \setminus \{(y,u)\}$

$$y^T H(x,s) + \Sigma_i (u^i)^T G^i(x,s) > x^T H(x,s) + \Sigma_i (s^i)^T G^i(x,s). \quad (12)$$

In the following we show that each evolutionary stable equilibrium is an attractor, and that all trajectories starting in $U \setminus \{(y,u)\}$, converge to the equilibrium in a 'well-behaved' manner.

Proposition 4.5. *Each evolutionary stable equilibrium is asymptotically stable.*

Proof. Let $(y,u) \in \Xi$ be an evolutionary stable equilibrium and let $U \subset \Xi$ be a neighborhood of (y,u) satisfying (12).

Let $V: \Xi \rightarrow \mathbb{R}$ be defined by $V(x,s) = (y - x)^T (y - x) + \Sigma_i (u^i - s^i)^T (u^i - s^i)$.

Obviously, $V(y,u) = 0$, and $V(x,s) > 0$ for all $(x,s) \in U \setminus \{(y,u)\}$.

Observe that $\partial V(x,s)/\partial x_i = -2(y_i - x_i)$ and $\partial V(x,s)/\partial s_h^i = -2(u_h^i - s_h^i)$ for all $i \in I^{n+1}$, $h \in I^{m+1}$. Hence,

$$\begin{aligned} dV(x,s)/dt &= \sum_i \partial V(x,s)/\partial x_i dx_i/dt + \sum_i \sum_h \partial V(x,s)/\partial s_h^i ds_h^i/dt = \\ &- 2 \sum_i (y_i - x_i) H_i(x,s) - 2 \sum_i \sum_h (u_h^i - s_h^i) G_h^i(x,s) = \\ &- 2 [(y - x)^T H(x,s) + \sum_i (u^i - s^i)^T G^i(x,s)] < 0 \text{ for all } (x,s) \in U \setminus \{(y,u)\} \text{ by Eq. (12).} \end{aligned}$$

Hence, V is a strict Lyapunov function on U , and (y,u) is asymptotically stable [cf. Hirsch and Smale (1974)]. \square

Rewriting Eq. (12) as $(y - x)^T H(x,s) + \sum_i (u^i - s^i)^T G^i(x,s) > 0$, we can interpret the part before the inequality, as the inner product of two $(n+1) \times (m+2)$ -dimensional vectors namely $((y-x)^T, (u^1-s^1)^T, \dots, (u^{n+1}-s^{n+1})^T)^T$ and $(H(x,s)^T, G^1(x,s)^T, \dots, G^{n+1}(x,s)^T)^T$. Then (12) implies that the angle between these vectors is always sharp for all (x,s) in $U \setminus \{(y,u)\}$. From this geometrical interpretation, the following may be derived.

Corollary 4.2. *The Euclidean distance to an evolutionary stable equilibrium decreases monotonically in time along each trajectory starting sufficiently close to it.*

For instance, an asymptotically stable equilibrium, where trajectories approach the equilibrium elliptically, is not an evolutionary stable equilibrium. This distinction between asymptotical and evolutionary stability is in line with e.g. Weissing (1990).

5. Strategic stability versus dynamic stability

The central concept in noncooperative game theory is the Nash-equilibrium, and it is well established in evolutionary game theory, that Nash-equilibria correspond with fixed points for the replicator dynamics. Nachbar (1991) has shown that if a trajectory under very general 'evolutionary' game dynamics converges from the interior of the strategy space, its limit point corresponds with a Nash-equilibrium. The second central concept in evolutionary game theory, the evolutionary stable strategy, was introduced by Maynard Smith and Price (1973). The evolutionary stable strategy is a Nash-equilibrium which is supposed to be stable against an

invasion of a small group employing a mutant strategy. However, the dynamical approach which seems to be implied by the very concept of evolutionary stability, is due to Taylor and Jonker (1978). This contribution introduced the replicator dynamics as selection dynamics in evolutionary game theory. Taylor and Jonker (1978) and Zeeman (1981) showed that evolutionary stability corresponds with asymptotical stability for the replicator dynamics, not vice versa. Bomze and Van Damme (1992) generalized these results for arbitrary fixed strategies combined with the replicator dynamics, while allowing the presence of more than one mutant strategy. Binmore and Samuelson (1992) remark that evolutionary equilibria should be able to repel several overlapping invasions of mutants. Swinkels (1992) argues that evolutionary equilibria should also be persistent against invading mutant strategies that are best replies for the strategic environment, which arises by their invasion of the system. It is now widely recognized that in essence, the original evolutionary stable strategy concept is a static concept, strategically stable against one invasion of mutants. True evolutionary stability requires an analysis of the dynamical system. Reconciling strategic stability and dynamic stability is not straightforward, as the remainder of this section may demonstrate.

Let $s \in S^m$, $u = (u^1, u^2, \dots, u^{n+1})$, then $u_{-i} \cup s = (u^1, \dots, u^{i-1}, s, u^{i+1}, \dots, u^{n+1})$. The strategy $x^i \in S^m$ is a best reply of subgroup i in (y, u) , if $x^i \in \{s \in S^m \mid s A^i S(y, u_{-i} \cup s) = \max_z \pi^i(y, u_{-i} \cup z), z \in S^m\}$, and write $x^i \in BR_i(y, u)$. Furthermore, for $x \in \prod_i S^m$ we write $x \in BR(y, u)$ if $x^i \in BR_i(y, u)$ for all $i \in I^{n+1}$. The state $(y, u) \in \Xi$ is a Nash-equilibrium if

$$u \in BR(y, u). \quad (13)$$

The notion of an evolutionary stable strategy formalizes that if an invasion of one group of mutants takes place, then the 'original' strategy fares better than the 'invading' strategy in the strategic environment that arises by this invasion. In our model, an invading group may consist of as many as $n+1$ subgroups playing $n+1$ possibly deviant strategies. Not all of these $n+1$ strategies must be different from the equilibrium strategies of the corresponding subgroups. A state (y, u) is an evolutionary stable strategy if $u \in BR(y, u)$, and additionally, if there exists a neighborhood $U \subset \Xi$ of (y, u) satisfying

$$0 < (u^i)^T g^i(x, s) \quad \text{if } s^i \neq u^i, \text{ for all } (x, s) \in U \setminus \{(y, u)\}. \quad (14)$$

Take namely, $s^i = \lambda u^i + (1-\lambda)z$, with $z \in S^m \setminus \{u^i\}$ arbitrary and $\lambda \in [0,1]$ sufficiently close to one such that $(x,s) \in U \setminus \{(y,u)\}$. Then it can be readily seen that (14) implies $(u^i - s^i)^T g^i(x,s) > 0$, which leads to $(1-\lambda)((u^i)^T g^i(x,s) - z^T g^i(x,s)) > 0$, hence $(u^i)^T g^i(x,s) > z^T g^i(x,s)$. The latter implies that the strategy u^i , which belongs to the evolutionary stable strategy, does better than any invading strategy z , against any population strategy $S(x,s)$, sufficiently similar to the equilibrium population strategy $S(y,u)$.

The Nash-equilibrium provides a 'prescription' as to what rational agents should play in a game, expecting all other agents to be rational as well. The evolutionary stable strategy is a refinement of the Nash-equilibrium. Equation (14) is called the stability condition in the literature [e.g. Van Damme (1991)]. To avoid confusion with the concept of dynamic stability of certain rest points of the dynamical system, we say that a point satisfying Eqs. (13) and (14), satisfies strategic stability. It is worthwhile to notice that in evolutionary models in general, and in this model in particular, these precepts of rationality are not met. The subgroups cannot deduce how to play a Nash-equilibrium, nor do they, even if they happen to play a Nash-equilibrium strategy, consciously play this strategy. Furthermore, the learning speeds are not infinitely large, which means that 'locating a better strategy' may be possible, but switching to this better alternative takes a nontrivial amount of time. In the meantime however, the strategic environment will have changed in general. An additional difficulty is that neither a Nash-equilibrium, nor an evolutionary stable strategy need to be a dynamic equilibrium in this model, since neither (13) nor (14) imply that $(u^i)^T A^i S(y,u) = (u^j)^T A^j S(y,u)$ for any pair $i,j \in I^{n+1}$, $i \neq j$. The latter would imply that $H_i(y,u) \neq 0$, or $H_j(y,u) \neq 0$, or both.

The state $(y,u) \in \Xi$ a saturated Nash-equilibrium if $u \in BR(y,u)$ and $f_i(y,u) \leq 0$ for all $i \in I^{n+1}$. One might think that each saturated Nash-equilibrium is also a saturated equilibrium, and indeed these equilibria often concur. The following propositions show that these equilibrium concepts do not always concur. The ensuing examples may suffice as proofs.

Proposition 5.1. *Let (y,u) be a saturated Nash-equilibrium, then $MBR(S(y,u)) \not\subseteq BR(y,u)$.*

Example 5.1. Let (y,u) be a saturated Nash-equilibrium satisfying $S(y,u) = (1/2, 1/2)^T$,

$$u^1 = (0,1)^T, y_1 = 0.2, \text{ and let}$$

$$A^1 = \begin{pmatrix} -0.1 & 1.3 \\ 0 & 1 \end{pmatrix}.$$

Then, $A^1 S(y,u) = (0.6, 0.5)^T$, hence $g^1_1(y,u) > g^1_2(y,u)$.

This implies that (y,u) is not a saturated equilibrium.

Proposition 5.2. *Let (y,u) be a saturated equilibrium, then $BR(y,u) \not\subseteq MBR(S(y,u))$.*

Example 5.2. Let (y,u) be a saturated equilibrium satisfying $S(y,u) = (1/2, 1/2)^T$,

$u^1 = (0, 1)^T$, $y_1 = 0.2$, and let

$$A^1 = \begin{pmatrix} 0.1 & -1.3 \\ 0 & -1 \end{pmatrix}.$$

Let $s = (1, 0)^T$, then it can be verified that $\pi^1(y,u) = -1/2 < \pi^1(y, u_{-i} \cup s) = -0.32$. Therefore (y,u) is not a saturated Nash-equilibrium.

A state $(y,u) \in \Xi$ is an evolutionary stable rest point if $u \in BR(y,u)$, and if there exists a neighborhood $U \subset \Xi$ of (y,u) satisfying for all $(x,s) \in U \setminus \{(y,u)\}$

$$y^T f(x,s) > 0 \text{ and } u^i \neq s^i \text{ implies } (u^i)^T g^i(x,s) > 0. \quad (15)$$

The following proposition states that if (15) holds in a neighborhood of (y,u) , this state must be a saturated equilibrium.

Proposition 5.3. *Each evolutionary stable rest point is a saturated equilibrium.*

Proof. Suppose (y,u) is an evolutionary stable rest point and (y,u) is not a saturated equilibrium. Then $i \in I^{n+1}$ exists such that $f_i(y,u) > 0$, or an $i \in I^{n+1}$ and an $h \in I^{m+1}$ exist such that $g^i_h(y,u) > 0$.

Suppose $i \in I^{n+1}$ exists such that $f_i(y,u) = \alpha > 0$.

Then, take (x,u) with $x = (1 - \lambda)y + \lambda e_i$, $\lambda \in [0, 1]$.

Since (17) implies $(y - x)^T f(x,s) > 0$ for $\lambda > 0$ sufficiently close to zero, it follows

that $\lambda y^T f(x,s) - \lambda e_i^T f(x,s) > 0$ for $\lambda > 0$ sufficiently small.

As $\lambda \downarrow 0$, we obtain $y^T f(x,s) \rightarrow 0$ by complementarity, whereas $e_i^T f(x,s) \rightarrow \alpha$ by our assumption. This establishes a contradiction.

Similarly, suppose $i \in I^{n+1}$ and $h \in I^{m+1}$ exist such that $g_h^i(y,u) = \alpha > 0$.

Take (y,s) with $s^i = (1 - \lambda)u^i + \lambda e_h$ for $\lambda \in [0,1]$, and $s^j = u^j$ for all $j \neq i$.

As $\lambda \downarrow 0$, we obtain $(u^i)^T g(y,s) \rightarrow 0$, whereas $(s^i)^T g(y,s) \rightarrow \alpha$, implying

$(u^i - s^i)^T g(y,s) < 0$, which contradicts (15). \square

Eq. (15) guarantees that an evolutionary stable rest point is isolated within the set of the saturated equilibria in U . Regrettably, Equation (15) does not imply dynamic stability.

The following establishes a connection between an evolutionary stable rest point and an evolutionary stable equilibrium in special cases. In general however, the relation between these equilibrium concepts is unclear. Let $(y,u) \in \Xi$ be a saturated equilibrium and let $P_{(y,u)}: \Xi \rightarrow \mathbb{R}^{n+1}$, be given by

$$P_{(y,u)}(x,s) = (y - x)^T (H(x,s) - f(x,s)) + \sum_i (u^i - s^i)^T (G^i(x,s) - g^i(x,s)). \quad (16)$$

This function can be used to show the connection between the different concepts of evolutionary stability as follows.

Proposition 5.4. *If for a saturated equilibrium (y,u) , there exists a neighborhood $U \subseteq \Xi$, such that $P_{(y,u)}(x,s) > 0$ for all $(x,s) \in U$, then if (y,u) is an evolutionary stable rest point, then (y,u) is an evolutionary stable equilibrium.*

Proof. Let (y,u) be an evolutionary stable rest point, and let $P_{(y,u)}(x,s) > 0$. This implies that $(y - x)^T (H(x,s) - f(x,s)) + \sum_i (u^i - s^i)^T (G^i(x,s) - g^i(x,s)) > 0$, which leads to $(y - x)^T H(x,s) + \sum_i (u^i - s^i)^T G^i(x,s) > (y - x)^T f(x,s) + \sum_i (u^i - s^i)^T g^i(x,s) = y^T f(x,s) + \sum_i (u^i)^T g^i(x,s) > 0$ by (15).

Hence, (y,u) is an evolutionary stable equilibrium. \square

6. Extensions of the model and discussion

Friedman (1991) and Nachbar (1990) advocate the use of more general dynamics and (relative) fitness functions. The motivation is that the evolutionary modelling approach is very well applicable in a wide range of social sciences and 'evolutionary' dynamics, as well as fitness functions, may be highly situation dependent. The first opportunity for generalization of our results, is along the line of Joosten (1993). In the latter paper, we examined the consequences of combining a general relative fitness function with population dynamics, which are weakly compatible with this relative fitness function. Generalizing the relative fitness function enables us to incorporate, for example, potential state dependent influences on the relative fitnesses of the subgroups in the population.

Let $\pi: \Xi \rightarrow \mathbb{R}^{n+1}$ be the payoff function and let this payoff function be continuous. Then, it is easy to see that the relative fitness function given by Eq. (2) satisfies continuity, as well as $x^T f(x,s) = 0$ for all $(x,s) \in \Xi$. Let furthermore, the continuous functions $\pi_h^i: \Xi \rightarrow \mathbb{R}$, $i \in I^{n+1}$, $h \in I^{m+1}$, denote the marginal contribution of pure action h at each state $(x,s) \in \Xi$, to the fitness of subgroup i . Observe that the equality $\pi^i(x,s) = \sum_k s_k^i \pi_k^i(x,s)$, need no longer hold in general. With regard to the marginal payoff functions given by Eq. (3), continuity and $(s^i)^T g^i(x,s) = 0$ for all $i \in I^{n+1}$, $(x,s) \in \Xi$, follow immediately. The population dynamics formalized in Eq. (4), are to be replaced by 'weakly compatible' population dynamics. Let $H: \Xi \rightarrow \mathbb{R}^{n+1}$, satisfy

- (f) H is continuous for all $(x,s) \in \Xi$,
- (g) $\sum_i H_i(x,s) = 0$ for all $(x,s) \in \Xi$,
- (h) $H_i(x,s) = 0$, whenever $x_i = 0$,
- (i) $x_i > 0$ implies $\text{sgn } H_i(x,s) = \text{sgn } f_i(x,s)$ for all $(x,s) \in \Xi$.

We call a function satisfying the restrictions above, weakly compatible with the relative fitness function. Let furthermore, the population dynamics be defined by the following system of $n+1$ (autonomous) differential equations

$$dx/dt = H(x,s), \quad \text{for all } (x,s) \in \Xi, \quad (17)$$

where the function $H: \Xi \rightarrow \mathbb{R}^{n+1}$ is weakly compatible with the relative fitness function. In the sequel, we call population dynamics given by Eq. (17) weakly compatible if the function H is weakly compatible with the relative fitness function. For population dynamics, the forward invariance property as captured in Restriction (h) is highly relevant. This restriction guarantees namely that no 'spontaneous' mutations occur in the model. In this respect, weakly compatible population dynamics and weakly compatible learning dynamics differ fundamentally. Note furthermore that the replicator dynamics are weakly compatible.

Clearly, the definitions of the saturated equilibrium by Eq. (6) and the evolutionary stable equilibrium by Eq. (12) need not be adapted. The Propositions 4.1, 4.2, 4.4, 4.5 and Corollary 4.1 will hold for this generalized dynamical system³. The following replaces Proposition 4.3.

Proposition 6.1. *Let (y,u) be the limit point for some trajectory $\{(x(t),s(t))\}_{t \geq 0}$, then the following properties hold:*

$$\begin{aligned}
 i \in C(y) \text{ implies} \quad h \in C(u^i): \quad & \pi_h^i(y,u) = \sum_k s_k^i \pi_k^i(y,u) \text{ and} \\
 & \pi^i(y,u) = \sum_j y_j \pi^j(y,u) \quad (18) \\
 h \in C(u^i): \quad & \pi_h^i(y,u) \leq \sum_k s_k^i \pi_k^i(y,u) \quad (19)
 \end{aligned}$$

$$\begin{aligned}
 i \in C(x(0)) \setminus C(y) \text{ implies} \quad h \in C(u^i): \quad & \pi_h^i(y,u) = \sum_k s_k^i \pi_k^i(y,u) \text{ and} \\
 & \pi^i(y,u) \leq \sum_j y_j \pi^j(y,u) \quad (20) \\
 h \in C(u^i): \quad & \pi_h^i(y,u) \leq \sum_k s_k^i \pi_k^i(y,u) \quad (21)
 \end{aligned}$$

$$i \notin C(x(0)) \text{ implies} \quad y_i = 0 \text{ and } u^i = s^i(0) \quad (22)$$

Hence, 'convergence implies saturated equilibrium' for trajectories from the interior of the state space, still holds. The results of Section 5 are not affected.

Arthur *et al.* (1986) show that general stochastic dynamics converge to the set of stable equilibria of the deterministic part of the dynamics under weak conditions, independent of the starting point of the dynamic process. Foster and Young (1990) analyze stochastic

³ We will not give these proofs here, as they constitute a repetition of the arguments used before.

evolutionary game dynamics, where the probability for a unique risk dominant equilibrium to be selected goes to unity, if the stochastic parts of the dynamics go to zero as time goes to infinity. Stronger results were obtained by Kandori *et al.* (1992), where a stochastic replicator-type learning process converges to a unique risk dominant equilibrium. Another extension would be to continually and stochastically disturb the payoffs as in e.g. Fudenberg and Harris (1992). Equilibrium selection is not the topic of this paper, and the learning process, as well as the population dynamics, may be path-dependent in the presence of multiple attracting equilibria. This in contrast to the contributions of e.g., Kandori *et al.* (1992), and Foster and Young (1990).

What may be the sources of these stochastic disturbances? Firstly, noise may be present in the payoff structure. For example, think of a biological system where temperature influences the fitnesses of the subgroups to different extents. Hence, on the short run the temperature may have a distorting influence on the evolutionary selection process. Obviously, these distortions should 'average out' in the long run. A second source of noise in the dynamical system may be 'mistakes' made in the learning process. Individuals may make mistakes in assessing which members of its subgroup to imitate, or which strategy to switch to, especially when the payoffs are similar. Individuals may even experiment to some extent. Furthermore, it should be noted that the dynamics used in evolutionary game theory, are approximations of very complicated stochastic matching processes. The approximation by deterministic differential equations is justified under a 'large number' assumption [Boylan (1992), Gilboa and Matsui (1992), Binmore *et al.* (1993)].

While we did not incorporate the influence of stochastic disturbances on the dynamical process into our model, we do acknowledge the relevance of such disturbances. The deterministic dynamics of our model are indeed in essence an approximation, the quality of which improves as the number of individuals in the population increases. However, one rather desirable consequence of the presence of very small stochastic disturbances, is that unstable or degenerate equilibria of the dynamical system are eliminated, as are unstable limit cycles for that matter. For predictive purposes only the asymptotically stable equilibria of the dynamical system are relevant as implied by Arthur *et al.* (1986).

To predict outcomes, one may wish to find dynamic equilibria. Firstly, we have demonstrated that trivial equilibria are relatively easily found, and some of these equilibria may indeed fulfill certain stability requirements as investigated in this paper. However,

finding or computing other equilibria, in particular equilibria in the interior of the state space, may be difficult, given the complex dynamical system. The problem of finding a saturated equilibrium, i.e. a state $(y,u) \in \Xi$ satisfying (6), for continuous functions $f:\Xi \rightarrow \mathbb{R}^{n+1}$, $g^i:\Xi \rightarrow \mathbb{R}^{m+1}$, $i \in I^{n+1}$, is called a nonlinear complementarity problem. To find saturated equilibria the variable dimension restart of Doup *et al.* (1987) may be employed. This simplicial algorithm may be started in an arbitrary point of the state space, which is a simplotope, i.e. the Cartesian product of a finite number of unit simplices. Such a variable dimension restart algorithm finds an arbitrarily accurate approximation of a saturated equilibrium in a finite number of steps. The conditions under which the algorithm accomplishes this, are precisely the conditions that guarantee the existence of at least one saturated equilibrium. In Joosten (1993) we have extensively argued that simulation involves a certain number of caveats which may lead to undesirable or inconclusive findings. The results of Saari and Simon (1978) and Saari (1985) suggest namely that one should rely on Scarf-type (1973), i.e. simplicial, methods to find equilibria. To the best of our knowledge, no algorithm exists that terminates exclusively with an approximation of a stable or an asymptotically stable equilibrium. Furthermore, if such an algorithm were to exist, it would not terminate at all in the cases where the dynamical system possesses no stable equilibrium.

7. Conclusions

'Standard' models in evolutionary game theory assume the strategies played by interacting subgroups in a population, to be genetically predetermined. This is well-suited when behavior is indeed completely or predominantly genetically determined. For more sophisticated beings however, this approach must be regarded as rather inept. Sophisticated beings may learn from their own experience (conditioning, reinforcement) or from the experience of other beings (imitation). Furthermore, knowledge may be transmitted directly in some form to offspring, or may be preserved in some form of collective memory.

We have therefore introduced the ability to learn from the strategic environment, for all subgroups in a population. Both the composition of the population and the strategies employed by the subgroups, are susceptible to changes. The subgroups (try to) improve their

fitness by gradually placing more weight on the pure strategies which yield higher payoffs. The population dynamics select the fitter subgroups over the less fit, meaning that fitter subgroups grow faster (i.e., have more offspring). Since all subgroups interact strategically and the strategic environment changes constantly, (truly) optimizing is generally an arduous task.

The dynamical system which we have introduced is generally rather complex, as it may be high-dimensional, and the differential equations describing the population dynamics and learning dynamics may be nonlinear. It is easy to show that certain (trivial) equilibria always exist, however it need not be that any trajectory under the dynamical system converges, let alone that any trajectory converges to any of these trivial equilibria. We have therefore introduced the saturated equilibrium and the evolutionary stable equilibrium. At a saturated equilibrium no subgroup has above average fitness, furthermore each subgroup in the population employs only a best-reply strategy against the population strategy. We have shown that a saturated equilibrium always exists, and that trajectories starting in the interior of the state space only converge to a saturated equilibrium. An evolutionary stable equilibrium is a saturated equilibrium which attracts all trajectories reaching a certain neighborhood of it. However, in contrast to the saturated equilibrium, an evolutionary stable equilibrium need not exist. We have shown the connections of the equilibrium concepts defined in this paper, with well-known dynamic equilibrium concepts. The dynamic properties of the saturated equilibrium and the evolutionary stable equilibrium suggest that these concepts are adequate generalizations of the Nash-equilibrium and the evolutionary stable state of the standard models in evolutionary game theory, respectively.

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