EVOLUTIONARY DYNAMICS
AND EQUILIBRIA[a]

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Abstract

Weakly compatible population dynamics describe changes in the composition of a population in a competitive environment, driven by differences in fitness levels within the population. We introduce the saturated equilibrium and the evolutionary stable equilibrium as (dynamic) generalizations of the Nash-equilibrium and the evolutionary stable strategy of evolutionary games. We examine the dynamic properties of these equilibrium concepts and relate them to equilibrium concepts from noncooperative game theory. The issues of existence, multiplicity, dynamical stability, and computability of equilibria are addressed. A variable dimension restart algorithm to solve the nonlinear complementarity problem may be used to compute a saturated equilibrium.

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1. Introduction

In evolutionary game theory, interacting groups within a population play genetically predetermined, distinguishable strategies. To each group a payoff is awarded, representing the fitness of this group. This payoff depends not only on the strategy employed by this group, but also on the strategies used by the other groups in combination with the distribution of the population over its subgroups. Fitness can be seen as a measure of potential to reproduce, and the evolution of the population share of each subgroup is axiomatically in accordance with its relative fitness.

Models in evolutionary game theory predominantly employ (bi)linear fitness functions, and assume the evolution of the composition of the population to be described by the so-called replicator dynamics. Recently, interesting contributions have appeared in the literature which use more general fitness functions or population dynamics. For instance, Hofbauer and Sigmund (1988) employ the replicator dynamics as selection dynamics for mathematical models derived from biological systems, which yield very general fitness functions. Nachbar (1990) and Friedman (1991) investigate general classes of 'evolutionary' dynamics, while the latter additionally admits general fitness functions.

What seems to motivate these contributions, is that the fitness function, as well as the dynamical process describing the evolution of the population, are strongly situation-dependent. An additional motivation for increasing the generality of these concepts, is the circumstance that evolutionary approaches are rapidly gaining popularity outside game theory or mathematical biology. In economics, for example, evolutionary dynamics are used to describe the evolution of the market shares of oligopolists, or to model the diffusion of technologies [e.g. Silverberg (1988)]. Game theory is used in other social sciences, where dynamical processes quite similar to evolutionary processes, may be employed to explain, for instance, the emergence of conventions as in Young (1993), or cooperation as in Axelrod (1984). Given this vast potential of applications, it is not clear beforehand that the replicator dynamics should be the only relevant evolutionary dynamics for all biological settings, let alone for all economical or social settings. Nor is it obvious why (bi)linear fitness functions should be the only type of fitness functions considered. Nevertheless, there seems to exist some reluctance to abandon (bi)linear fitness functions and replicator dynamics.

We generalize the approaches Nachbar (1990) and Friedman (1991) in the line of

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Hofbauer and Sigmund (1988). Our aim is to develop concepts which are meaningful for very general 'evolutionary' settings, yet come close to established results, when applied to standard evolutionary games. We assume that all strategical interactions, as well as all other relevant influences, are fully captured by a continuous function, namely the relative fitness function. A relative fitness function attributes for every state to each subgroup in the population the difference between the subgroup's fitness and the population share weighted average fitness of the population. By definition, a relative fitness function satisfies complementarity. We assume furthermore that the population dynamics are weakly compatible with the relative fitness function. Weakly compatible dynamics formalize that the change in population share of any nonextinct subgroup corresponds in sign with the relative fitness of this subgroup.

Our main equilibrium concepts are the saturated equilibrium and the evolutionary stable equilibrium. At a saturated equilibrium each subgroup with positive population share has highest (relative) fitness. We prove existence of at least one saturated equilibrium for arbitrary relative fitness functions, and demonstrate that each saturated equilibrium is a rest point for weakly compatible dynamics. If any trajectory starting in the interior of the state space converges, its limit point is necessarily a saturated equilibrium. Our analysis implies furthermore that any stable equilibrium is a saturated equilibrium, whereas any equilibrium which is not a saturated equilibrium, is unstable for all weakly compatible dynamics. The evolutionary stable equilibrium is defined in terms of local properties of the population dynamics. While each evolutionary stable equilibrium is an asymptotically stable equilibrium, the converse statement need not hold. It is namely proven that any trajectory reaching a certain neighborhood of an evolutionary stable equilibrium, converges towards it and the Euclidean distance to the equilibrium strictly decreases in time as the trajectory approaches it. A saturated equilibrium is shown to correspond with a Nash-equilibrium in standard models in evolutionary game theory. The dynamic properties of the evolutionary stable equilibrium suggest that this concept is an adequate dynamic pendant of the evolutionary stable strategy.

We address the topics of multiplicity, stability and computability of equilibria of weakly compatible dynamics for arbitrary relative fitness functions, as well as for less general settings. We derive conditions on the dynamical system for uniqueness of an interior saturated equilibrium, which imply either asymptotical stability, or instability of this equilibrium. The dynamical system is generally nonlinear, and even low-dimensional nonlinear dynamical systems may display an astonishing degree of complexity. Hence, researchers interested in

finding saturated equilibria may encounter severe computational problems. We propose therefore the following procedure. All vertices of the unit simplex are checked, which generally is fairly easy to accomplish from a computational point of view. To find other saturated equilibria a variable dimension restart algorithm to solve the so-called nonlinear complementarity problem, may be employed. Such an algorithm may be started in an arbitrary point of the unit simplex, and converges to an arbitrarily accurate approximation of a saturated equilibrium for arbitrary relative fitness functions. A variable dimension restart algorithm accomplishes this feat under precisely the conditions which guarantee existence of at least one saturated equilibrium, namely continuity and complementarity of the relative fitness function. The algorithm of Doup *et al.* (1987) seems the most efficient in this class of algorithms for the problem studied.

This paper is organized as follows. In Section 2, we give a brief overview of related work in evolutionary game theory and economics, dynamical systems, and equilibrium programming. We formulate the model in Section 3. In Section 4, we define the saturated equilibrium and the evolutionary stable equilibrium and we compare these concepts with other equilibrium concepts, dynamic as well as game-theoretical. In Section 5, we study conditions on the dynamical system which have consequences on the stability and the multiplicity of dynamic equilibria. We focus on some fundamental problems connected with finding equilibria in Section 6, and give the procedure to find equilibria, which may include an algorithm to approximate saturated equilibria. Section 7 concludes.

2. Related literature

Maynard Smith and Price (1973) have introduced the central concept in evolutionary game theory, the evolutionary stable strategy. An evolutionary stable strategy is a Nash-equilibrium which satisfies an additional strategic stability requirement. This stability requirement aimes to formalize that when the population is invaded by any mutant strategy, this mutant strategy performs worse than the original equilibrium strategy, in the strategic environment that arises by its invasion, provided that the new strategic environment is sufficiently similar to the equilibrium strategic environment. Consequently, the mutant strategy is supposed to be driven out and the equilibrium strategy is to prevail in the long run. It is widely recognized that the

evolutionary stable strategy is an essentially static concept, not a dynamic concept as the idea behind it would suggest. Real evolutionary stability requires a dynamic stability concept.

Not until Taylor and Jonker (1978), did the replicator dynamics appear in evolutionary game theory to describe the evolution of the composition of a population. Taylor and Jonker (1978) and Zeeman (1981) investigate relations of the evolutionary stable strategy with several dynamic equilibrium concepts for the replicator dynamics. They show that every evolutionary stable strategy is an asymptotically stable state of the replicator dynamics, and that not every asymptotically stable strategy is an evolutionary stable strategy. Though alternative evolutionary dynamics have been proposed [cf. Nachbar (1990), Friedman (1991)], the replicator dynamics virtually monopolize the literature. Hofbauer and Sigmund (1988) give an extensive treatment of the replicator dynamics for general relative fitness functions. Hirsch and Smale (1974) are our main point of reference with regard to nonlinear differential equations and dynamical systems.

Evolutionary economics is the next of kin to evolutionary game theory in economics, as it too employs a biological paradigm. Evolutionary economics also uses replicator(-type) dynamics as selection dynamics, e.g. Nelson and Winter (1982), Iwai (1984), and Silverberg (1988). The main conceptual difference between evolutionary economics and evolutionary game theory is that in the latter discipline the number of strategies present is fixed. In evolutionary economics, new strategies are perpetually appearing as new technologies or innovations, 'mutations' in the biological metaphor, which takes these models out of the framework of the analysis of this paper[1]. Regardless namely of the evolutionary dynamics employed, the dynamical system never settles down in the long run. On the other hand, not every invading mutant need to have a significant

impact on the dynamical system. Hence, temporarily, i.e. in the short run, the system might very well display inertia or convergence to a certain outcome, until a new invading mutation has an impact on the system forceful enough to upset this situation. In this context, David (1985) even argues succesfully that a technically inferior strategy may drive out superior strategies for prolonged periods of time.

Fixed points of evolutionary dynamics can be associated with equilibria in noncooperative game theory and economics. In a bounded rationality framework, replicator-

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[1] Our attention to this point was drawn by Richard Nelson, who commented on an earlier version of this paper.

type dynamics are interpreted as learning processes, or processes to motivate equilibrium selection[2]. For a large class of evolutionary dynamics, the vertices of the state space are fixed points. Unfortunately, these fixed points may be dynamically unstable, and finding interior fixed points or stable fixed points, may be difficult. This is a discouraging prospect since precisely the stable equilibria of dynamics are relevant for predictions. Doup *et al.* (1987) provide an efficient algorithm to solve the nonlinear complementarity problem, which may be used to find a saturated equilibrium. Van der Laan and Talman (1987) provide a survey on variable dimension restart algorithms.

3. The evolutionary model

Consider a population which consists of n+1 distinguishable, interacting subgroups. This interaction of the subgroups has consequences on their respective abilities to reproduce. Fitness is a measure of ability to reproduce, determined by (genetically given) behavior in combination with the composition of the population. Since behavior is genetically predetermined, fitness can be analyzed as depending solely on the composition of the population, implying fitness to be a function of the population shares of all subgroups. Let $I^{n+1} = \{1,2,...,n+1\}$, and let $x = (x_1, x_2, ..., x_{n+1})^T \in S^n = \{y \in \mathbb{R}^{n+1} | \sum_j y_j = 1, y_i \geq 0 \text{ for all } i \in I^{n+1}\}$ denote a vector of population shares, henceforth to be called a state. Let $E: S^n \to \mathbb{R}^{n+1}$ be a continuous function, attributing to every subgroup its fitness at each state. We assume that all stategic interaction, as well as all possible other influences on the fitness of the subgroups, is fully captured by this function. We suppress time-notations whenever confusion seems unlikely. The inner product of vectors $x, y \in \mathbb{R}^{n+1}$, is denoted by $x^T y$. For $x, y \in \mathbb{R}^{n+1}$ the Euclidean distance is denoted by $d_2(x,y) = [\sum_i (y_i - x_i)^2]^{1/2}$. Furthermore, e_j, $j \in I^{n+1}$, is the vertex $x \in S^n$ with $x_j = 1$. For $z \in \mathbb{R}^{n+1}$, we may write for example, $z \geq \mathbf{0}$, by which we mean $z_i \geq 0$ for all $i \in I^{n+1}$. Proofs may be found in the Appendix. We denote the cardinality of a set S by $|S|$.

States change over time subject to evolutionary dynamics, which means that if a subgroup has above-average (below-average) fitness at a certain state, its population share

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[2] See e.g. Crawford (1985), Gilboa and Matsui (1991), or Kandori *et al.* (1992).

increases (decreases). The <u>relative fitness function</u> $f:S^n \to \mathbb{R}^{n+1}$ is given by

$$f_i(x) = E_i(x) - \sum_j x_j E_j(x) \qquad \text{for all } i \in I^{n+1}, x \in S^n. \tag{1}$$

The relative fitness function attributes for every state, to each subgroup, the difference between the fitness of the subgroup and the average fitness of the population, weighted by the population shares of all subgroups. To model changes in the composition of the population, we consider a dynamic process represented by the following system of n+1 (autonomous) differential equations:

$$dx/dt = G(x) \qquad \text{for all } x \in S^n, \tag{2}$$

where $dx/dt = (dx_1/dt, dx_2/dt, ..., dx_{n+1}/dt)^T = (G_1(x), G_2(x), ..., G_{n+1}(x))^T$, denotes the continuous-time change of the population share of each subgroup. This function G is independent of the time variable, and depends entirely on the composition of the population. Furthermore, this function must fulfill the criterium of weak compatibility. Given a relative fitness function f, the function $G:S^n \to \mathbb{R}^{n+1}$, is <u>weakly compatible</u> if

      a)   G is continuous,

      b) $\sum_i G_i(x) = 0$ for all $x \in S^n$,

      c) $x_i = 0$ implies $G_i(x) = 0$ for all $x \in S^n$, $i \in I^{n+1}$, and

      d) $x_i > 0$ implies sign $G_i(x)$ = sign $f_i(x)$ for all $x \in S^n$, $i \in I^{n+1}$.

For every $y \in \mathbb{R}$, sign y is defined as: sign y = +1 if y > 0, sign y = 0 if y = 0, sign y = −1 if y < 0. <u>Dynamics are weakly compatible</u> if the function G in (2) is weakly compatible. No additional restrictions are placed on G in this section. Our definition of weak compatibility differs from Friedman's (1991). Most notably, we do not require piecewise differentiability of G. Furthermore, it should be noted that Friedman defines the dynamics on the Cartesian product of the strategy-spaces of all subgroups, whereas our dynamics are defined on the state space formed by the population shares of the subgroups. The replicator dynamics are represented by $dx/dt = (x_1 f_1(x), ..., x_{n+1} f_{n+1}(x))^T$. It is easy to verify that these dynamics fulfill the criteria for weak compatibility.

A function, say $h: Z \to D$ with $Z \subseteq D$, satisfies <u>complementarity</u> if $x^T h(x) = 0$ for all $x \in Z$. It can be verified that *for every relative fitness function $f: S^n \to \mathbb{R}^{n+1}$ and for all $x \in S^n$, it follows that $x^T f(x) = 0$.* Hence, for every evolutionary setting the relative fitness function satisfies continuity and complementarity. By reasoning conversely, the following question arises. Given a function $g: S^n \to \mathbb{R}^{n+1}$, satisfying continuity and complementarity, can we construct an evolutionary setting such that the resulting relative fitness function is equal to g? The answer is affirmative, and the argument is straightforward. Take $g: S^n \to \mathbb{R}^{n+1}$, satisfying continuity and complementarity. Let a population consist of $n+1$ subgroups, and define the fitness of each subgroup $i \in I^{n+1}$, at every state $x \in S^n$, by $E_i(x) = g_i(x) + c$, where $c \in \mathbb{R}$. The corresponding relative fitness function f satisfies

$$f_i(x) = (g_i(x) + c) - \Sigma_j\, x_j\, (g_j(x) + c) = g_i(x) \text{ for all } i \in I^{n+1}, x \in S^n. \qquad \textbf{(2)}$$

This exercise suggests that *a relative fitness function is characterized by continuity and complementarity.* To the best of our knowledge there do not exist any additional restrictions on (relative) fitness functions, which would limit the candidates for such functions to a smaller class [see also Hofbauer and Sigmund (1988)].

The latter reminded us of some classical results in general equilibrium theory. Sonnenschein (1972, 1973), Mantel (1974), and Debreu (1974)[3] demonstrate that any continuous function satisfying complementarity can be approximated arbitrarily closely by an (aggregate or community) excess demand function derived from a pure exchange economy with utility maximizing agents. In the same context, a Walrasian tâtonnement is a dynamic process where the change in time of the (relative) price of any good is a sign preserving function of the excess demand for that good, provided that all prices are strictly positive [cf. Uzawa (1961)]. If any price becomes equal to zero, boundary restrictions (should) prevent the process from reaching negative prices. It is well established that excess demand functions for pure exchange economies are homogeneous of degree zero in the prices. It is therefore permitted to restrict the analysis of the relative fitness functions, as well as Walrasian tâtonnement processes, to the unit simplex, i.e. the subset of the all nonnegative prices adding up to unity. The parallels between excess demand functions and relative fitness functions on

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[3] Eric van Damme recommended to prove a characterization in the spirit of Sonnenschein, Mantel and Debreu. Hans Peters and the author found the argument given.

one hand, and Walrasian tâtonnement processes and weakly compatible dynamics on the other, are quite striking in our opinion.

We end this section with some examples. The first example is fairly standard [cf. Bomze and Van Damme (1992)]. The second example is a 'delinearized' variant of the first. These examples should not lead to the misconception that our approach is only applicable to settings which generate such simple relative fitness functions. For instance, in economics the relevant strategic variables may be (mixes) of variables as prices, quantities of goods produced, investments allocated, or advertisement expenditures.

Example 3.1. Let $A = \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$, and let $s^1 = (\tfrac{1}{2},\tfrac{1}{2})^T$, $s^2 = (2/3,1/3)^T$, $s^3 = (1/4,3/4)^T$. We define the fitness functions $E_i:S^2 \to \mathbb{R}^3$ for all $i \in I^3$, by $E_i(x) = (s^i)^T A (\Sigma_{j=1,2,3} x_j s^j)$.

Hence, $E_1(x) = \tfrac{1}{2}$

$E_2(x) = \tfrac{1}{2} x_1 + 5/9\, x_2 + 5/12\, x_3$

$E_3(x) = \tfrac{1}{2} x_1 + 5/12\, x_2 + 5/8\, x_3.$

Then the relative fitness functions are

$f_1(x) = \tfrac{1}{2} - \tfrac{1}{2}x_1(1+x_2+x_3) - 5/9\,(x_2)^2 - 5/6x_2x_3 - 5/8\,(x_3)^2.$

$f_2(x) = -\tfrac{1}{2}x_1(x_2+x_3) + 5/9\, x_2(1-x_2) + 5/12x_3(1-x_2) - 5/8\,(x_3)^2.$

$f_3(x) = -\tfrac{1}{2}x_1(x_2+x_3) - 5/9\,(x_2)^2 + 5/12x_2(1-x_3) + 5/8\, x_3(1-x_3).$

Example 3.2. Let A, s^1, s^2 and s^3 be as in Example 3.1, let $E_i:S^2 \to \mathbb{R}^3$ for all $i \in I^3$ be defined by $E_i(x) = (s^i)^T A (\Sigma_{j=1,2,3} x_j s^j) + (2-x_i)^{-1}$.

Then,

$f_1(x) = \tfrac{1}{2} - \tfrac{1}{2}x_1(1+x_2+x_3) - 5/9\,(x_2)^2 - 5/6x_2x_3 - 5/8\,(x_3)^2 +$
$\qquad (1-x_1)(2-x_1)^{-1} - x_2(2-x_2)^{-1} - x_3(2-x_3)^{-1}.$

$f_2(x) = -\tfrac{1}{2}x_1(x_2+x_3) + 5/9\, x_2(1-x_2) + 5/12x_3(1-x_2) - 5/8\,(x_3)^2 -$
$\qquad x_1(2-x_1)^{-1} + (1-x_2)(2-x_2)^{-1} - x_3(2-x_3)^{-1}.$

$f_3(x) = -\tfrac{1}{2}x_1(x_2+x_3) - 5/9\,(x_2)^2 + 5/12x_2(1-x_3) + 5/8\, x_3(1-x_3) -$
$\qquad x_1(2-x_1)^{-1} - x_2(2-x_2)^{-1} + (1-x_3)(2-x_3)^{-1}.$

The parts distinguishing the fitness functions of Example 3.2 from those in Example 3.1, may

be regarded as state-dependent feedbacks. Arthur (1988), David (1985), and Dosi *et al.* (1993), offer motivations for feedbacks in economics. Generally speaking, if there exist increasing (decreasing) returns to scale, or if there exist externalities, positive (negative) state dependent feedbacks may be expected.

4. Equilibria of weakly compatible dynamics

The state $y \in S^n$ is a <u>fixed point</u> if $G(y) = \mathbf{0}$ in Equation (3). At a fixed point changes in the composition of the population come to a rest. The fixed point $y \in S^n$ is <u>stable</u> if for any (open) neighborhood $U \subset S^n$ of y, a neighborhood $V \subseteq U$ of y exists such that any trajectory starting in V remains in U. The stable fixed point $y \in S^n$ is <u>asymptotically stable</u> if additionally a neighborhood $W \subset S^n$ of y exists such that all trajectories starting in a point of W converge towards y. Hirsch and Smale (1974) treat several methods to examine the dynamical stability of fixed points. We use Lyapunov's second method on several occasions, only partly because of its considerable elegance. An additional reason for employing Lyapunov's method is technical. This method does namely not rely on the (partial) differentiability of the relative fitness function or of the weakly compatible dynamics. Lyapunov's method can furthermore be employed to examine the dynamical properties of sets of points [cf. Uzawa (1961)].

Existence of as many fixed points as there are subgroups in the population, is (trivially) guaranteed, since *for weakly compatible dynamics, each vertex of the unit simplex is a fixed point*. Given a relative fitness function f, the state $y \in S^n$ is a <u>saturated equilibrium</u>[4] if $f(y) \leq \mathbf{0}$. A saturated equilibrium y is <u>strict</u> if $f_j(y) = 0$ for precisely one $j \in I^{n+1}$. At a saturated equilibrium each group with positive population share has highest fitness. Any group with below-average-fitness has population share equal to zero. The following propositions pertain to arbitrary relative fitness functions and weakly compatible dynamics.

Finding $y \in S^n$ for which $z(y) \leq \mathbf{0}$ for a continuous function $z:S^n \rightarrow \mathbb{R}^{n+1}$ satisfying complementarity, is called a <u>nonlinear complementarity problem with respect to z</u>. Finding

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[4] In constrast to Hofbauer and Sigmund (1988) differentiability of the relative fitness function is not assumed.

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a saturated equilibrium for a relative fitness function f, is a nonlinear complementarity problem with respect to f.

Proposition 4.1. *There exists at least one saturated equilibrium.*

Proposition 4.2. *Every saturated equilibrium is a fixed point.*

It follows straightforwardly that each fixed point located in the interior of the unit simplex is a saturated equilibrium. Since an unstable interior equilibrium is also a saturated equilibrium, it is evident that not every saturated equilibrium is stable. The following propositions show further connections between equilibrium concepts.

Proposition 4.3. *Every strict saturated equilibrium is asymptotically stable.*

Proposition 4.4. *Every stable equilibrium is a saturated equilibrium.*

The proof of Proposition 4.4 leads to the following corollary.

Corollary 4.1. *No fixed point in the boundary of the unit simplex that is **not** a saturated equilibrium, is reached by any trajectory from the relative interior of the unit simplex.*

The state $y \in \mathbb{R}^{n+1}$, is a <u>limit point</u> if there exists a trajectory $\{z(t)\}_{t \geq 0} \subset S^n$, such that $G(z(0)) \neq \mathbf{0}$ and $\mathrm{Lim}_{t \to \infty} z(t) = y$. From the continuity of weakly compatible dynamics combined with the compactness of S^n, it follows that y is a fixed point and that $y \in S^n$. By Corollary 4.1, each limit point for a trajectory starting in the relative interior of the unit simplex, located in a boundary of the unit simplex, must be a saturated equilibrium. Hence, *the set of limit points for trajectories starting in the interior of the unit simplex is a subset of the set of saturated equilibria*. Trajectories starting in a boundary of the unit simplex, never leave this boundary, and if such a trajectory converges, the limit point need not be a saturated equilibrium. However, the limit point of such a trajectory, is a saturated equilibrium with regard to the restriction of the relative fitness function to this lower-dimensional subset. It is not clear what the connection between stable equilibria and limit points, is in general. See also Figure 1.

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A fixed point $y \in S^n$, is an <u>evolutionary stable equilibrium</u> if and only if there exists a neighborhood $U \subset S^n$ of y satisfying

$$y^T G(x) > x^T G(x) \qquad \text{for all } x \in U \setminus \{y\}. \tag{4}$$

Let ESE be the set of all evolutionary stable equilibria. Let furthermore ASE and SNTE denote the sets of all asymptotically stable equilibria and strict saturated equilibria respectively. The connections between SNTE, ESE and ASE are given in the following proposition.

Proposition 4.5. *SNTE $\subseteq$ ESE $\subseteq$ ASE.*

In the proof of this proposition, we use the function $V(x) = -\frac{1}{2} \Sigma_i (y_i - x_i)^2$, and show that this function is a strict Lyapunov function for a neighborhood of y, the evolutionary stable equilibrium. Since, obviously $V(x) = -\frac{1}{2} (d_2(x,y))^2$, part (a) of the proof of Proposition 4.5, leads to the following.

Corollary 4.2. *The Euclidean distance to an evolutionary stable equilibrium decreases monotonically in time along any trajectory starting sufficiently close to it.*

A geometrical interpretation of Equation (4) is that the angle between $(y - x)$ and $G(x)$ for all $x \in U \setminus \{y\}$, is always sharp. This also follows (indirectly) from Corollary 4.2, since if for any trajectory $\{x(t)\}_{t \geq 0}$ at any point in time t^* the angle between $(y - x(t^*))$ and $G(x(t^*))$ would be equal to or more than 90 degrees, then there would exist $t' > t^*$ satisfying $d_2((x(t'),y) \geq d_2(x(t^*),y)$.
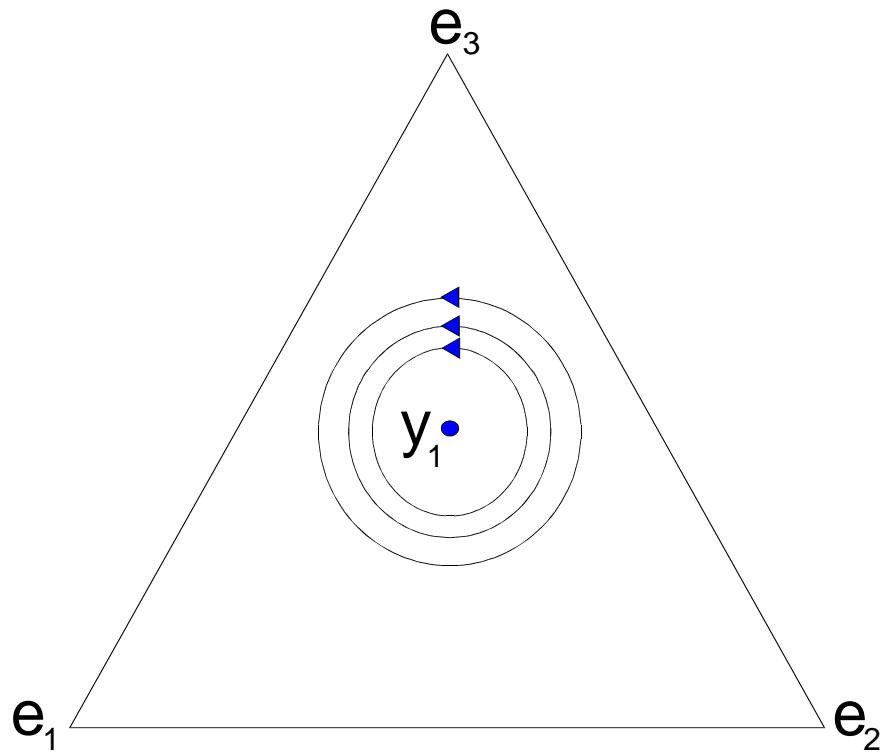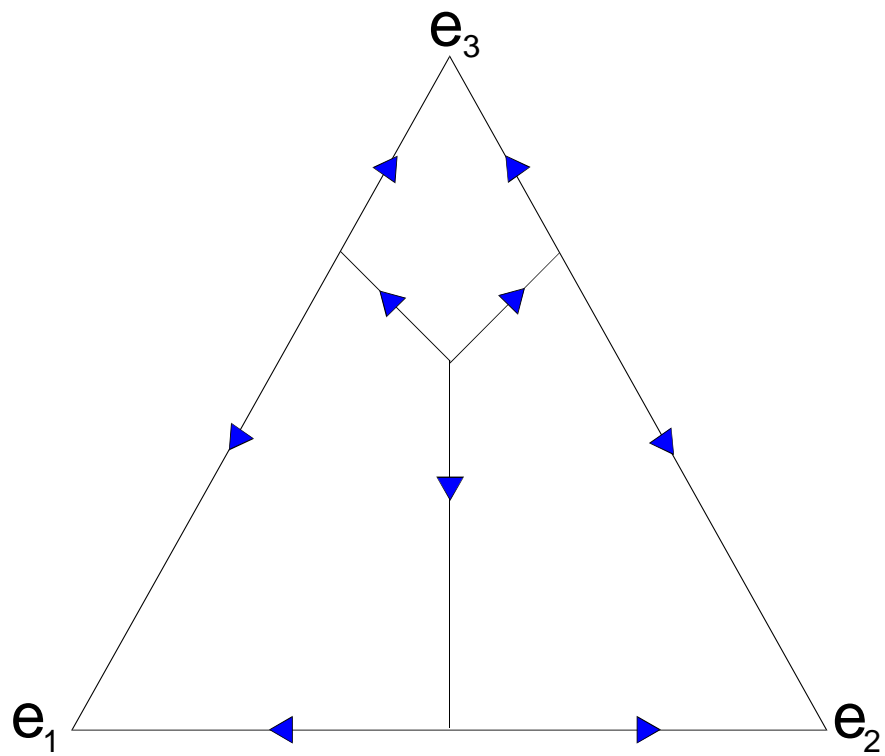
Figure 1. (a) y_1 is a stable equilibrium, not asymptotically stable, (b) y_1, y_2, y_3 are unstable limit points.

If a boundary of the unit simplex acts as a repellor no saturated equilibrium exists on this boundary. Furthermore, there exists a positive threshold value such that certain subgroups have population shares at least equal to this value in the long run, provided these subgroups had nonzero population shares initially. The following formalizes this.

Proposition 4.6. *If $k \in K \subseteq I^{n+1}$ implies $f_k(x) > 0$ whenever $x_k = 0$, then $\delta > 0$ exists, such that for all trajectories $\{x(t)\}_{t \geq 0}$: $\lim_{t \to \infty} \inf x_k(t) \geq \delta$ provided $x_k(0) > 0$, for all $k \in K$.*

Proposition 4.6 implies that if the system is subjected to an incidental shock smaller than δ, then no group in such a set K will become extinct. We do not model such shocks, but one can think of real-life systems where a shock is caused by an environmental catastrophe or an invasion. Furthermore, each saturated equilibrium y, satisfies $y_k \geq \delta$ for all $k \in K$. If the condition of Proposition 4.6 holds for $K = I^{n+1}$, this is sufficient for <u>permanence</u> of the system [cf. Hofbauer and Sigmund (1988)]. Permanence of a system implies furthermore that no group of invading mutants is driven out, which does not necessarily mean that the dynamical system is in steady turmoil. The system may, for example, possess an asymptotically stable equilibrium, and return to it after each invasion by a small group of mutants.

Taylor and Jonker (1978) and Zeeman (1981) connect the Nash-equilibrium and the evolutionary stable strategy, with dynamic equilibrium concepts for the replicator dynamics. Following their approach, we examine several cases where the relative fitness function $f : S^n \to \mathbb{R}^{n+1}$ is given by

$$f_k(x) = e_k^T A x - x^T A x \text{ for all } x \in S^n, k \in I^{n+1}, \tag{5}$$

where A is an $(n+1) \times (n+1)$-matrix. For the symmetric bimatrix game (A, A^T), the mixed strategy combination (y,y), $y \in S^n$, is a <u>symmetric Nash-equilibrium</u> if $x^T A y \leq y^T A y$ for all $x \in S^n$. Furthermore, $y \in S^n$ is an <u>evolutionary stable strategy</u> if $x \in S^n$, then $x^T A y \leq y^T A y$, and additionally if $x \neq y$ and $x^T A y = y^T A y$, then $x^T A x < y^T A x$. It is well established that every symmetric bimatrix game possesses a symmetric Nash-equilibrium, that every symmetric Nash-equilibrium of a symmetric bimatrix game is a fixed point of the replicator dynamics, that every strict symmetric Nash-equilibrium is an evolutionary stable strategy, and that every evolutionary stable strategy is an asymptotically stable equilibrium

for the replicator dynamics [cf. Van Damme (1991)]. The following establishes the connections between the saturated equilibria of (5), and the Nash-equilibria of the bimatrix game (A,A^T).

Proposition 4.7. *The following two statements are equivalent:*

        i) y is a saturated equilibrium of the relative fitness function given by (5),

        ii) (y,y) is a Nash-equilibrium of the bimatrix game (A,A^T).

Since saturated equilibria for (5) correspond to symmetric Nash-equilibria of the bimatrix game (A,A^T), the following observations are in order. Any Nash-equilibrium is a fixed point for weakly compatible dynamics for (5). Furthermore, our analysis implies that any limit point for a trajectory starting in an interior nonequilibrium point and governed by weakly compatible dynamics for the relative fitness function (5), corresponds to a Nash-equilibrium of (A,A^T). This generalizes a result in Nachbar (1990). If a fixed point of the weakly compatible dynamics is stable, it must correspond to a Nash-equilibrium. These observations indicate that the saturated equilibrium is a suitable generalization of the Nash-equilibrium.

Van Damme (1991) characterizes an evolutionary stable strategy in a symmetric bimatrix game (A,A^T). This characterization 'translates' into our notations as follows. The strategy $y \in S^n$ is an evolutionary stable strategy, if and only if for a relative fitness function given by (5), there exists a neighborhood $U \subseteq S^n$ of y such that

$$y^T f(x) > 0 \qquad \text{for all } x \in U \backslash \{y\}. \qquad \textbf{(6)}$$

Note the similarities between Equations (4) and (6), where the functions f and G are perfectly interchangeable, since it should be noted that $x^T f(x) = 0$ for all $x \in S^n$. The geometrical interpretation of (6) is that the angle between the vectors $(y - x)$ and $f(x)$ in $U \backslash \{y\}$ is always sharp. However, the vector $f(x) \in \mathbb{R}^{n+1}$ at $x \in S^n$, may point outward of the unit simplex, hence it is not possible to employ the function f as population dynamics in Equation (3).

If (6) holds in a neighborhood $U \subseteq S^n$ of a saturated equilibrium y for an **arbitrary** relative fitness function f, then it follows easily that no saturated equilibrium exists in $U \backslash \{y\}$. In the following we call a state satisfying (6) for an arbitrary relative fitness function in a reduced neighborhood of this state, a <u>generalized evolutionary stable state</u>. However, since

asymptotical stability is not implied by (6) for arbitrary relative fitness functions, we think the generalized evolutionary stable state not to be an 'adequate' generalization of the evolutionary stable strategy. Proposition 4.5 and Corollary 4.2 suggests that *the evolutionary stable equilibrium is a suitable dynamic generalization of the evolutionary stable state* for arbitrary relative fitness functions and weakly compatible dynamics. Friedman's (1991) evolutionary equilibrium is equivalent to an asymptotically stable state. Let y be a saturated equilibrium and let $H_y : S^n \to \mathbb{R}^{n+1}$, be given by

$$H_y(x) = (y - x)^T (G(x) - f(x)). \tag{7}$$

Proposition 4.8. *If for a saturated equilibrium y there exists a neighborhood $U \subseteq S^n$, such that $H_y(x) < (>) 0$ for all $x \in U$, then 'y is an evolutionary stable equilibrium' implies 'y is a generalized evolutionary stable state' ('y is a generalized evolutionary stable state' implies 'y is an evolutionary stable equilibrium').*

We conclude this section with an analysis of Example 3.1.

Example 3.1 (continued).

The vertices e_2 and e_3 are strict saturated equilibria, hence evolutionary stable equilibria, since it can be checked that $f_1(e_2) = -1/18$, $f_3(e_2) = -5/36$, $f_1(e_3) = -1/8$, $f_2(e_3) = -5/24$.

Convex combinations of e_1 and $(0,3/5,2/5)^T$, are saturated equilibria, since it can be confirmed that $f_1(\lambda e_1 + (1-\lambda)(0,3/5,2/5)^T) = f_2(\lambda e_1 + (1-\lambda)(0,3/5,2/5)^T) = f_3(\lambda e_1 + (1-\lambda)(0,3/5,2/5)^T) = 0$.

In Figure 2, Example 3.1 is illustrated. The line segment connecting e_1 and $(0,3/5,2/5)^T$ separates the basins of attraction of e_3 and e_2. Trajectories starting above (below) this line segment, converge to e_3 (e_2) under all weakly compatible dynamics. No trajectory traverses this line segment as all of its points are equilibria. The basin of attraction of e_3 is larger than the basin of attraction of e_2. All other equilibria have basins of attraction with measure zero.
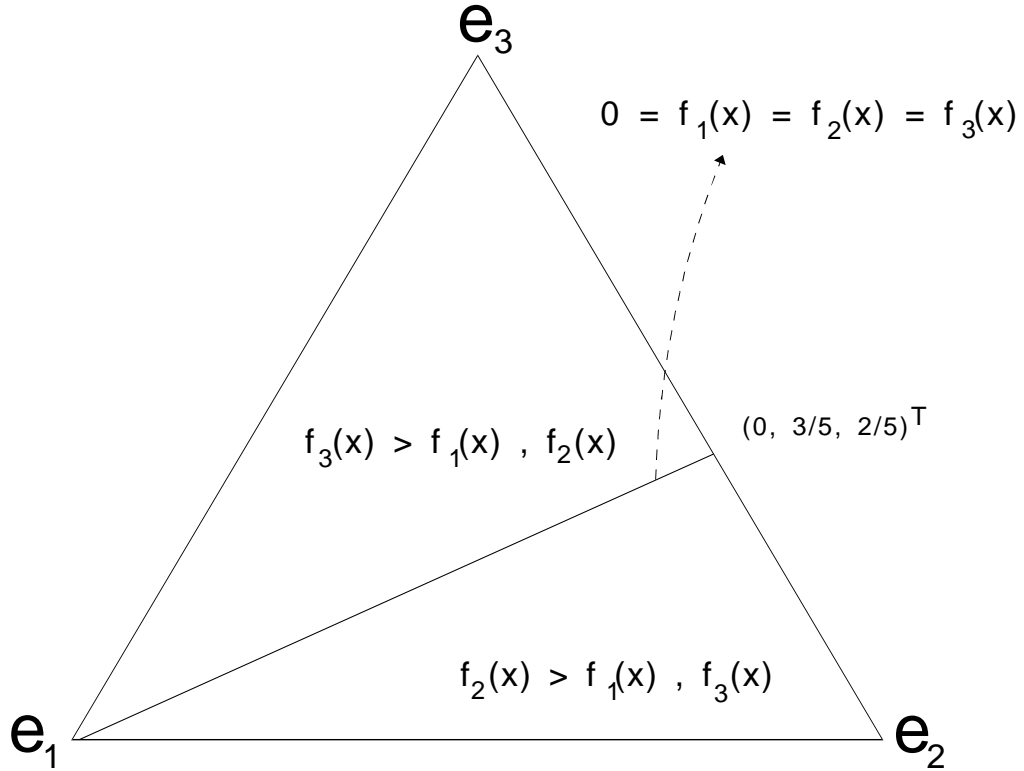
Figure 2. The three vertices are saturated equilibria. All convex combinations of e_1 and $(0,3/5,2/5)^T$ are saturated equilibria. Note that the set of saturated equilibria consists of compact subsets of S^2. The vertices e_2 and e_3 are strict saturated equilibria, which implies that they are evolutionary stable equilibria.

5. Multiplicity and stability of equilibria of weakly compatible dynamics

Since each vertex of the unit simplex is a fixed point for weakly compatible dynamics, it follows easily that the number of fixed points is at least equal to the number of subgroups in the population. Generically, however, each of the $2^{n+1} - 2$ faces of the n-dimensional unit simplex contains an equilibrium, though not necessarily a saturated one. We have shown that at least one saturated equilibrium exists for arbitrary relative fitness functions. Proposition 4.5 implies $|SNTE| \leq |ESE| \leq |ASE|$, whereas furthermore $|SNTE| \leq n+1$. In Example 3.1, there exist two isolated saturated equilibria, whereas all convex combinations of e_1 and $(0,3/5,2/5)^T$ are saturated equilibria, illustrating that the set of equilibria consists of compact subsets of the unit simplex. Note that if the condition of Proposition 4.6 is fulfilled for a set with cardinality one, all dynamic equilibria in the facet opposite vertex corresponding with

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the index in this set, are not saturated, therefore unstable. For sets with cardinality greater than one, none of the vertices of the unit simplex is a saturated equilibrium.

The first setting where where more precise statements can be made about the number of equilibria, is the situation where all saturated equilibria are regular, meaning that the determinant of the Jacobian matrix of the relative fitness function at such a point exists and is nonzero. A well-known result is that if all saturated equilibria are regular, their number is odd [cf. Dierker (1972), Hofbauer and Sigmund (1988)]. Furthermore, Hofbauer and Sigmund (1988) call a system of which the population dynamics are given by a differentiable function $h:\mathbb{R}^{n+1}\rightarrow\mathbb{R}^{n+1}$, cooperative (competitive) on a set $G \subseteq \mathbb{R}^{n+1}$, if $\partial h_i(x)/\partial x_j \geq (\leq) 0$ for all $i \neq j$, $x \in G$. We call a system <u>strongly cooperative</u> (<u>competitive</u>) if for a differentiable relative fitness function f,

$$\partial f_i(x)/\partial x_j > (<) 0 \text{ for all } i \neq j \text{ and all } x \in \text{ int } S^n. \qquad \textbf{(8)}$$

Arrow *et al.* (1959) consider the case that all goods in an economy are strong gross substitutes. Mathematically, strong cooperativeness can be viewed as gross substitutability, where in (8) $f:S^n\rightarrow\mathbb{R}^{n+1}$ is an excess demand function and $x \in S^n$ is a vector of normalized prices. The following lemma is well-known in mathematical economics, and we employ this lemma in the proof of the ensuing proposition.

Lemma 5.1. *If all commodities are strongly gross substitutes, the excess demand function satisfies homogeneity of degree zero, and there exists a positive equilibrium price vector, then this price vector is uniquely (up to a scalar multiple) determined.*

Proposition 5.1. *For a strongly cooperative system, if there exists an interior equilibrium, then it is the unique interior equilibrium.*

For a strongly competitive evolutionary system with relative fitness function f, the system with relative fitness function −f is strongly cooperative. Proposition 5.1 shows that if an interior equilibrium y exists for the system with relative fitness function −f, then y is the unique interior equilibrium. Hence, y is the unique interior equilibrium for f.

The following lemma is a straightforward application of a theorem of Uzawa (1961)

to the present framework.

Lemma 5.2. *If, for a strongly cooperative system, there exists an interior equilibrium, then each trajectory of weakly compatible dynamics starting in the interior of S^n, converges to it.*

If the conditions of Lemma 5.2 are fulfilled, then the interior equilibrium is asymptotically stable. Conversely, if a system is strongly competitive, the interior equilibrium is unstable. If strong cooperativeness (competitiveness) is weakened by $\partial f_i(x)/\partial x_j \geq (\leq) 0$ for all $i \neq j$ and for all $x \in S^n$ in (8), then uniqueness of the interior equilibrium need not hold [cf. Uzawa (1961)]. However, all trajectories for weakly compatible dynamics starting in the interior of S^n converge to some interior equilibrium.

If a strongly cooperative system is permanent, then Proposition 4.6 implies that every saturated equilibrium lies in the interior of S^n, whereas Proposition 5.1 implies that there exists only one equilibrium in the interior of S^n. By Lemma 5.2 this point is asymptotically stable, hence $|\text{NTE}| = |\text{ASE}| = 1$, where $|\text{NTE}|$ denotes the number of saturated equilibria. If, for a strongly competitive system, it holds that $f_i(x) < 0$ whenever $x_i = 0$, then Proposition 4.3 implies that every vertex of S^n is an asymptotically stable equilibrium. Proposition 5.1 and Lemma 5.2 show that exactly one interior equilibrium exists, which is unstable, and therefore $|\text{NTE}| > |\text{ASE}| \geq |\text{SNTE}| = n+1$. We conclude this section with a qualitative analysis of Example 3.2.

Example 3.2. (continued)

First we compute all partial derivatives $\partial f_i(x)/\partial x_j$ for $i \neq j$.

$\partial f_1(x)/\partial x_2 = -\tfrac{1}{2}x_1 - (10/9)x_2 - (5/6)x_3 - 1/(2-x_2) - x_2(2-x_2)^{-2}$,

$\partial f_1(x)/\partial x_3 = -\tfrac{1}{2}x_1 - (5/6)x_2 - (10/8)x_3 - 1/(2-x_3) - x_3(2-x_3)^{-2}$,

$\partial f_2(x)/\partial x_1 = -\tfrac{1}{2}(x_2+x_3) - 1/(2-x_1) - x_1(2-x_1)^{-2}$,

$\partial f_2(x)/\partial x_3 = -\tfrac{1}{2}x_1 + (5/12)(1-x_2) - (10/8)x_3 - 1/(2-x_3) - x_3(2-x_3)^{-2}$,

$\partial f_3(x)/\partial x_1 = -\tfrac{1}{2}(x_2+x_3) - 1/(2-x_1) - x_1(2-x_1)^{-2}$,

$\partial f_3(x)/\partial x_2 = -\tfrac{1}{2}x_1 - (10/9)x_2 + (5/12)(1-x_3) - 1/(2-x_2) - x_2(2-x_2)^{-2}$.

It is obvious that $\partial f_1(x)/\partial x_2$, $\partial f_1(x)/\partial x_3$, $\partial f_2(x)/\partial x_1$, $\partial f_3(x)/\partial x_1$ are negative for all $x \in S^n$. Furthermore, note that

$\partial f_2(x)/\partial x_3 \quad = -\tfrac{1}{2}x_1 + (5/12)(1-x_2) - (10/8)x_3 - 1/(2-x_3) - x_3(2-x_3)^{-2}$

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$$= -\tfrac{1}{2}(x_1+x_2+x_3) + 5/12 + (1/12)x_2 - ((10/8)-\tfrac{1}{2})x_3 - 1/(2-x_3)$$
$$\qquad - x_3(2-x_3)^{-2}$$
$$= -(1/12)(1-x_2) - (3/4)x_3 - 1/(2-x_3) - x_3(2-x_3)^{-2}.$$

Similarly, we obtain

$\partial f_3(x)/\partial x_2 \qquad = -(1/12)(1-x_3) - (11/18)x_3 - 1/(2-x_2) - x_2(2-x_2)^{-2}.$

This implies that the evolutionary system is **strongly competitive**.

Since $f_2(e_1) = f_3(e_1) = -\tfrac{1}{2}$, $f_1(e_2) = -5/9$, $f_3(e_2) = -23/46$, $f_1(e_3) = -5/8$, $f_2(e_3) = -17/24$, the three vertices of the unit simplex are evolutionary stable equilibria for arbitrary weakly compatible dynamics.

By Proposition 5.1 there exists at most one interior equilibrium, and Lemma 5.2 states that this equilibrium is unstable for weakly compatible dynamics.
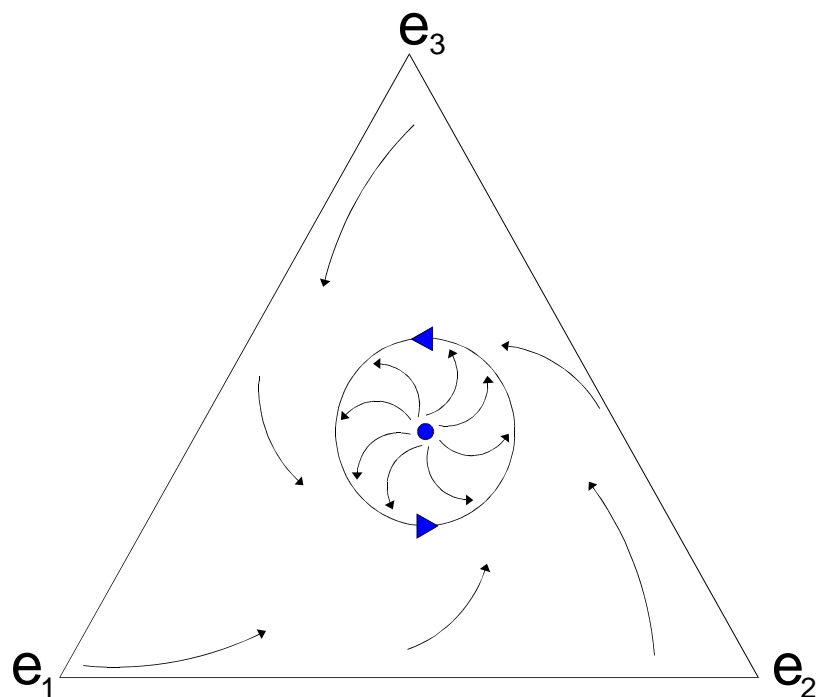
If we restrict the relative fitness function to any of the facets of the unit simplex, then this yields a strongly competitive system on this facet. By Proposition 5.1 there exists at most one interior equilibrium on this facet, whereas by Lemma 5.2 this equilibrium is unstable.

Concluding, there exist three evolutionary stable equilibria, three equilibria on each of the facets of the unit simplex, and at most one interior equilibrium.

6. Finding equilibria of weakly compatible dynamics

Simulation is quite popular in the evolutionary branches of game theory and economics. The most common practice is to use discrete-time variants of the continuous-time dynamics as an approximation of the latter. Simulation may indeed provide some insights in the properties of the dynamical system. However, simulation in order to find equilibria may be unpractical for a number of reasons. For large numbers of subgroups in a population, the simulated dynamics are less tractable and computational efforts generally become quite considerable. An additional difficulty is that weakly compatible dynamics become 'slow' near the boundaries of the unit simplex. It might therefore seem as if a trajectory converges to a point on such a boundary, while the real limit point is located elsewhere. This phenomenon is termed *pseudo-convergence* in Nachbar (1992). Furthermore, there exist problems of a more fundamental nature. Dekel and Scotchmer (1992) show that simulated dynamics

approximating discrete-time replicator dynamics, may spiral outward to the boundaries of the unit simplex [see also Weissing (1990)]. Related to this, is the problem of *cycling*, i.e., a certain sequence of points is repeated infinitely, without reaching an equilibrium. Firstly, detecting that the simulated dynamics cycle requires a large memory size of the computer, since it may take many iterations before a sequence of points repeats itself. Secondly, there is the problem that simulated dynamics may cycle, while the real continuous-time dynamics do not. For generalized Rock-Scissors-Paper games, where continuous-time replicator dynamics admit *stable limit cycles* only for populations with four or more subgroups, Weissing (1990) shows existence of stable limit cycles already for three subgroups for discrete-time replicator dynamics. Finally, we have generalized the relative fitness functions and the population dynamics considerably, hence existence of *strange attractors* must be anticipated (see Figure 3). It is unknown what the behavior of the discrete-time dynamics in the presence of strange attractors, is in general. Therefore, although the rest points of continuous-time and discrete-time dynamics concur, their qualitative behaviors may be very different. Hence, conclusions about (non−) convergence of continuous dynamics should not be based on simulated dynamics, which are discrete by conception.

Figure 3. Adaptation from Varian (1978). Example due to Scarf (1960).

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The force driving weakly compatible evolutionary dynamics is the relative fitness function, which fulfills the requirements of an excess demand function for a pure exchange economy, namely continuity and complementarity. The results of Sonnenschein (1972,1973), Mantel (1974) and Debreu (1974) imply that problem of finding a saturated equilibrium is (mathematically) equivalent to finding an economic equilibrium. Many methods have been suggested in the literature to find economic equilibria, which may provide solutions to find saturated equilibria for **some** relative fitness functions [e.g., Smale (1976)]. However, the results of Saari and Simon (1978) and Saari (1985) imply that, to find saturated equilibria for **arbitrary** relative fitness functions, one should rely on Scarf-type algorithms [Scarf (1973)].

One such algorithm is the variable dimension restart algorithm of Doup *et al.* (1987). This simplicial algorithm is globally convergent, which means that it converges to an approximation of a saturated equilibrium for arbitrary relative fitness functions while being started from an arbitrary point of the unit simplex. The algorithm employs a simplicial subdivision of the unit simplex depending on the starting point of the algorithm. A piecewise linear path in a sequence of adjacent simplices of variable dimension is generated, connecting the starting point with an approximation of a saturated equilibrium. Whenever accuracy is not satisfactory, the algorithm is restarted in this approximating solution using a finer simplicial subdivision and a more accurate approximation is generally found. This $(2^{n+1}-2)$-ray algorithm of Doup *et al.* (1987) seems to be the most efficient in the class of variable dimension restart algorithms. We think that describing a variable dimension restart algorithm in detail would go beyond the scope of this paper. For an illustration we refer to Figure 4.
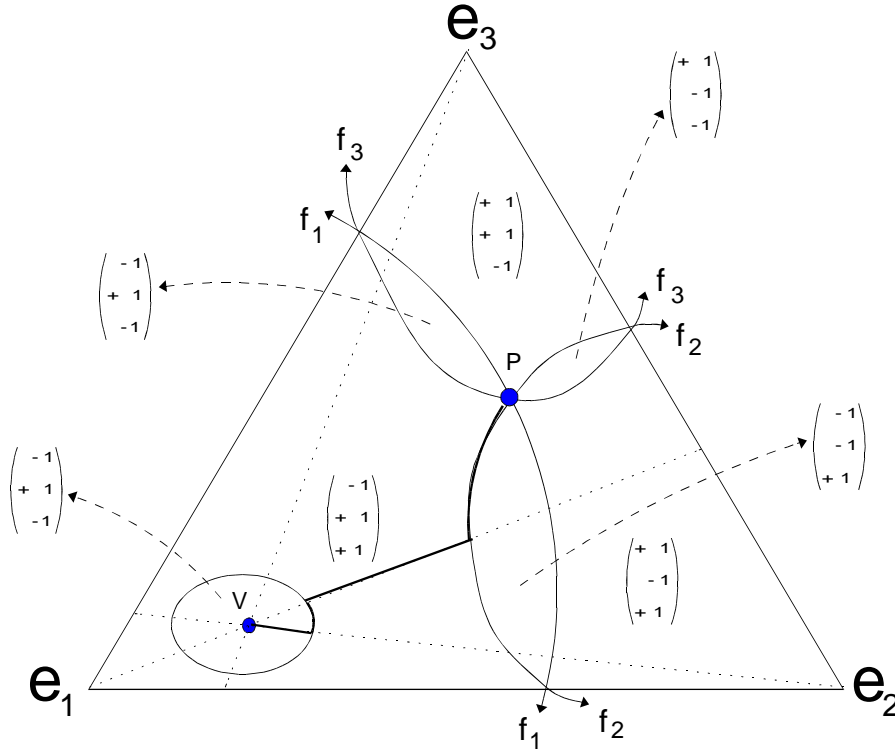
Figure 4. The algorithm approximately follows the heavy curve leading from the starting point v to the saturated equilibrium p. The rays leaving v are indicated by broken lines. The vector $(+1,-1,-1)^T$ denotes a set where (sign $f_1(x)$, sign $f_2(x)$, sign $f_3(x))^T = (+1,-1,-1)^T$; $f_i=0$ denotes a curve where $f_i(x)=0$.

7. Conclusions and discussion

We investigated an evolutionary model, while relaxing several assumptions with respect to fitness functions and population dynamics. For this purpose, we introduced the relative fitness function, which attributes for every state to each subgroup in a population its fitness relative to the average fitness. In deriving the majority of our results, we have only required continuity of this relative fitness function, which satisfies complementarity by definition. We specified a class of evolutionary population dynamics, called weakly compatible dynamics, under which the change in the population share of each nonextinct subgroup corresponds in sign with the relative fitness of this subgroup. We introduced the saturated equilibrium, showed that the set of saturated equilibria is nonempty for arbitrary relative fitness functions, and that each saturated equilibrium is a fixed point for weakly compatible dynamics. At a

saturated equilibrium, every subgroup with positive population share has highest fitness. Additionally, an evolutionary stable equilibrium was defined, having very attractive dynamic properties. Every trajectory reaching a certain neighborhood of an evolutionary stable equilibrium converges towards it, while the Euclidean distance to the equilibrium strictly decreases monotonically. For traditional evolutionary games, a saturated equilibrium corresponds with a symmetric Nash equilibrium. The dynamic properties of the evolutionary stable equilibrium suggest that this concepts is an adequate dynamic generalization of the well-known evolutionary stable strategy.

There exist striking mathematical similarities between the relative fitness functions of evolutionary approaches used in game theory, and other social sciences on one hand, and the excess demand functions for pure exchange economies on the other. Both type of functions satisfy continuity and complementarity. Furthermore, a saturated equilibrium corresponds with an economic equilibrium, whereas weakly compatible dynamics for a given relative fitness function correspond with a Walrasian tâtonnement process for an excess demand function of the same mathematical form (see also Table I). A parallel to Sonnenschein (1972, 1973), Mantel (1974) and Debreu (1974), stating that continuity and complementarity are the properties which characterize excess demand functions, seems to exist for relative fitness functions. Hence, as is widely recognized in general equilibrium theory, we should anticipate **any** dynamic process on the unit simplex as the outcome of weakly compatible dynamics combined with **some** relative fitness function. It is well known in economics, that multiple equilibria exist even in 'simple' cases [cf. Kehoe (1988)], and that tâtonnement-type dynamics based on excess demand functions, possess strange attractors such as stable limit cycles [cf. Scarf (1960)].

The price to pay for the increased generality of our model, is therefore that finding or computing equilibria with certain stability requirements may be quite hard. We propose a two-step method to find saturated equilibria. First all vertices of the unit simplex are examined. Each strict saturated equilibrium is asymptotically stable and these equilibria are necessarily vertices of the state space. To find other, in particular interior, saturated equilibria, variable dimension restart algorithms provide a solution. A variable dimension restart algorithm designed for the nonlinear complementarity problem, converges to an arbitrarily accurate approximation of a saturated equilibrium within a finite number of steps for arbitrary relative fitness functions. Research indicates that the algorithm of Doup *et al.* (1987) is the

most efficient variable dimension restart algorithms. Having approximated a saturated equilibrium, further analysis may be undertaken. The conditions for an evolutionary stable equilibrium are readily checked, whereas checking other types of stability generally involves a rather elaborate analysis. It should be noted that equilibria which are 'refinements' of the saturated equilibrium need not exist in general. Hence, if there were to exist an algorithm that terminates exclusively with such an equilibrium, it may not terminate at all in some cases.

Evolutionary game theory	Pure exchange economies with normalized prices
Relative fitness function	Excess demand function
Complementarity of the relative fitness function	Walras' Law
Saturated equilibrium (Nash equilibrium)	Economic equilibrium
Weakly compatible dynamics	Walrasian tâtonnement process
Permanence	Desirability of all goods
Cooperative system	'Gross substitutes'-case
Evolutionary Stable Strategy	(Local) 'revealed preferences' at equilibrium
Strategic versus evolutionary stability	Hicksian versus Samuelsonian stability

Table I.   Mathematical similarities shown, found or exploited in this paper.

We see an extension of our approach in the rapidly expanding field of adaptive learning. The dynamics employed in this paper, weakly compatible dynamics, can be interpreted as adaptive learning dynamics in a bounded rationality context to formalize inductive learning. For learning dynamics however, the condition which guarantees forward invariance, Condition c, is not mandatory. It may consequently be replaced by a weaker version. If weakly compatible dynamics converge from the interior of the state space, they converge towards a point which possesses properties, which are similar to those of a Nash-equilibrium.

Our approach may be used to investigate settings in biology or economics where rather complex relative fitness functions and dynamics are required. Neither any nonlinearities of the relative fitness function or the weakly compatible dynamics, nor high dimensionality of the dynamical system, pose unsurmountable problems. Existence of at least one saturated equilibrium remains guaranteed, and it may be found by the method proposed.

We have argued repeatedly in this paper that there exist mathematical similarities between certain concepts in seemingly distant subdisciplines of economics. The literature on excess demand functions for pure exchange economies and price adjustment dynamics derived from these excess demand functions, is vast. It may prove fruitful to examine the applicability of certain price adjustment processes as plausible alternative evolutionary or learning dynamics. For instance, Nikaidô (1959) introduced a tâtonnement process described by the so-called Brown-Von-Neumann differential equations, and (re)introducing a variant of these dynamics in game theory as evolutionary dynamics seems to offer one such plausible alternative.

APPENDIX

Proof 4.1:   Let F be the point-to-set mapping from S^n to the subsets of S^n, for every $x \in S^n$ defined by

$$F(x) = \text{conv}(\{e_j \mid f_j(x) = \max_i f_i(x), i \in I^{n+1}\}).$$

Then F is upper-semicontinuous, and for every $x \in S^n$ the set $F(x)$ is nonempty, convex and compact. From Kakutani (1974), it follows that there exists $y \in S^n$, satisfying $y \in F(y)$.

Let $c = \max_i f_i(y)$ and let $T = \{k \mid f_k(y) = c\}$. Since $y \in F(y)$ it follows that nonnegative numbers λ_j, $j \in T$, summing up to one, exist, such that

$$y = \Sigma_{j \in T} \lambda_j e_j.$$

Since $y_j = 0$ for all $j \notin T$, we obtain:

$$0 = y^T f(y) = \Sigma_{j \in T} y_j f_j(y) + \Sigma_{j \notin T} y_j f_j(y) = \Sigma_{j \in T} y_j f_j(y) = \Sigma_{j \in T} \lambda_j e_j c = c.$$

Consequently, $\max_i f_i(y) = 0$, $i \in I^{n+1}$, which proves the proposition.   ■

Proof 4.2:   Let $y \in S^n$ satisfy $f(y) \leq \mathbf{0}$. Then it follows by complementarity that for every $j \in I^{n+1}$ precisely one of the following situations occurs

(a) $y_j > 0$ and $f_j(y) = 0$,

(b) $y_j = 0$ and $f_j(y) \leq 0$.

Both imply $G(y) = \mathbf{0}$ by definition.   ■

Proof 4.3: Let $y \in S^n$ be a strict saturated equilibrium.

By definition there exists some $j \in I^{n+1}$ satisfying $0 = f_j(y) > \max_{i \neq j} f_i(y)$,

then $0 = \Sigma_i \, y_i \, f_i(y) = y_j \, f_j(y) + \Sigma_{i \neq j} \, y_i \, f_i(y) = \Sigma_{i \neq j} \, y_i \, f_i(y) \leq$

$\Sigma_{i \neq j} \, y_i \, \max_{i \neq j} f_i(y) = (1 - y_j) \max_{i \neq j} f_i(y) \leq 0$. This implies $y_j = 1$.

Since f is continuous, an $\in \, > 0$ and a $\delta > 0$ exist, satisfying $f_i(x) < -\in$ for

all $i \neq j$ and all $x \in U = \{u \in S^n | \, 1 - u_j < \delta\}$. Then, complementarity

implies $f_j(x) = - \Sigma_{i \neq j} \, x_i \, f_i(x)/x_j > 0$ for all $x \in U \backslash \{y\}$.

Furthermore, $G(y) = \mathbf{0}$ by Proposition 4.2, whereas $x \in U \backslash \{y\}$ implies

$$\text{sign } G_i(x) \leq 0 \text{ for all } i \neq j, \tag{9}$$

$$\text{sign } G_j(x) > 0. \tag{10}$$

Let $V: U \rightarrow \mathbb{R}$ be defined by $V(x) = (x_j - 1) - \Sigma_{i \neq j} \, x_i$ for all $x \in U$.

Clearly, $V(y) = 0$ and $V(x) < 0$ for all $x \in U \backslash \{y\}$. Since $\partial V(x)/\partial x_j = 1$ and

$\partial V(x)/\partial x_i = -1$, $i \neq j$, $dV(x)/dt = \Sigma_k \, \partial V(x)/\partial x_k \, G_k(x) = G_j(x) - \Sigma_{i \neq j} \, G_i(x)$.

Hence, $dV(y)/dt = 0$, and (9) and (10) imply $dV(x)/dt > 0$ for all

$x \in U \backslash \{y\}$. Therefore, V is a strict Lyapunov-function on U, and y is

asymptotically stable. ∎

Proof 4.4: Suppose y is a stable equilibrium and y is not a saturated equilibrium.

Then $f_j(y) > 0$ and $y_j = 0$ for some $j \in I^{n+1}$. Since f is continuous, $\in \, > 0$

and $\delta > 0$ exist such that $f_j(x) > \in$ for all $x \in U$, where $U = \{u \in S^n |$

$\max_i |u_i - y_i| < \delta, \, i \in I^{n+1}\}$.

Let $\{z(t)\}_{t \geq 0}$ be a trajectory with $z(0) \in U$ and let $z_j(0) = \alpha > 0$. Since

$z_j(0) = \alpha$ and $f_j(z(0)) > \in$, it follows that sign $G_j(z(0)) = +1$, implying

$G_j(z(0)) > 0$. Furthermore, $z_j(t) > z_j(0)$ and $G_j(z(t)) > 0$ while

$\{z(t)\}_{t \geq 0} \subset U$. Then $\{z_j(t)\}_{t \geq 0}$ increases monotonically while $\{z(t)\}_{t \geq 0} \subset U$.

Hence, there exists $t^* > 0$ such that $\max_i |z_i(t^*) - y_i| = \delta, \, i \in I^{n+1}$, thus

$z(t^*) \notin U$. This leads to a contradiction as for U no neighborhood $V \subseteq U$

of y exists such that a trajectory $\{z(t)\}_{t \geq 0}$ with $z(0) \in V$ and $z_j(0) > 0$

remains in U. ∎

Proof 4.5: a) ESE $\subseteq$ ASE.

Let y be an evolutionary stable equilibrium and let $U \subset S^n$ be the

neighborhood of y where (4) holds. Let $V:S^n \to \mathbb{R}$ be given by

$V(x) = -\frac{1}{2} \Sigma_i (y_i - x_i)^2$, then $V(y) = 0$, and $V(x) < 0$ for all $x \in U\backslash\{y\}$.

Furthermore, $dV(y)/dt = 0$ and $dV(x)/dt = \Sigma_i \partial V(x)/\partial x_i \, dx_i/dt =$

$\Sigma_i (y_i - x_i) G_i(x) = y^T G(x) - x^T G(x) > 0$ for all $x \in U\backslash\{y\}$. This means that

V is a strict Lyapunov-function on U, hence y is asymptotically stable.

b) SNTE $\subseteq$ ESE.

Let $y = e_j$, $j \in I^{n+1}$ be a strict saturated equilibrium, then from the proof

of Proposition 4.3, it follows that there exists a neighborhood U such that

(9) and (10) hold in $U\backslash\{y\}$. Hence, $x \in U\backslash\{y\}$ implies $y_j - x_j > 0$, and

$y_i - x_i \leq 0$ for all $i \neq j$. Therefore, $(y_j - x_j) G_j(x) > 0$, as well as

$(y_i - x_i) G_i(x) \geq 0$ for all $i \neq j$, $x \in U\backslash\{y\}$. Hence, $(y - x)^T G(x) > 0$ for all

$x \in U\backslash\{y\}$. ∎

Proof 4.6: Let $K \subseteq I^{n+1}$ satisfy $f_k(x) > 0$, whenever $x_k = 0$ for all $k \in K$. Let $S(K) =$

$\{x \in S^n \mid x_k = 0$, for some $k \in K\}$.

Since f is continuous, there exists $\delta_k > 0$ for each $k \in K$, such that $x_k < \delta_k$

implies $f_k(x) > 0$. Now, take $0 < \delta < \min \{\delta_k \mid k \in K\}$, and $U = \{u \in S^n \mid$

$u_k \leq \delta$ for all $k \in K\}$, and let $V:U \to \mathbb{R}^{n+1}$ be given by $V(x) = \min_{k \in K} x_k$ for

all $x \in U$.

Clearly, $V(x) = 0$ for all $x \in S(K)$, $V(x) > 0$ for all $x \in U\backslash S(K)$.

Furthermore $dV(x)/dt = \Sigma_j \partial V(x)/\partial x_j \, dx_j/dt = G_{i*}(x)$, where $i* \in K$ is some

index satisfying $x_{i*} = \min_{k \in K} x_k$. Therefore, if $x \in S(K)$, then $G_{i*}(x) = 0$, and

if $x \in U\backslash S(K)$, then $G_{i*}(x) > 0$.

Hence $dV(x)/dt = 0$ for all $x \in S(K)$, whereas $dV(x)/dt > 0$ for all

$x \in U\backslash S(K)$, implying that V is a strict Lyapunov-function on U. Hence,

$S(K)$ acts as a repellor for each trajectory $\{z(t)\}_{t\geq 0}$ with $z(0) \in U\backslash S(K)$,

since $\{V(z(t))\}_{t\geq 0}$ increases strictly while $\{z(t)\}_{t\geq 0} \subset U\backslash S(K)$.

Therefore, $t* \geq 0$ exists, satisfying $V(z(t*)) = \delta$, hence the trajectory leaves

U. This completes the proof, as no trajectory reaches U from $S^n\backslash U$. ∎

Proof 4.7: Let y be a saturated equilibrium for the relative fitness function f

given by (5). Since $y_j > 0$ leads to $f_j(y) = 0 = \max_i f_i(y)$, it follows that

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$y_j > 0$ implies $e_j^T A y = \max_i e_i^T A y$.

Furthermore, $e_j^T A y < \max_i e_i^T A y$ implies $y_j = 0$, which in turn implies (y,y) is a Nash equilibrium of the bimatrix game (A,A^T).

Conversely, let (y,y) be a Nash equilibrium of the bimatrix game (A,A^T).

Then, $y_j > 0$ implies $e_j^T A y = \max_i e_i^T A y$.

Let $T = \{j \in I^{n+1} | e_j^T A y = \max_i e_i^T A y\}$, then $y^T A y = \Sigma_k y_k e_k^T A y$ $= \Sigma_{k \in T} y_k (\max_i e_i^T A y) = \max_i e_i^T A y$. This immediately implies $f(y) \leq \mathbf{0}$ for the relative fitness function f given by (5). ■

Proof 4.8:   Let y be a saturated equilibrium, let $H_y : S^n \to \mathbb{R}^{n+1}$ be given by (7), and let $U \subseteq S^n$ be a neighborhood of y, such that $H_y(x) > 0$ for all $x \in U$.

Then $H_y(x) = (y - x)^T (G(x) - f(x)) > 0$, therefore $(y - x)^T G(x) > (y - x)^T f(x) = y^T f(x)$. Hence, (6) implies (4). By reverting all inequalities, we obtain the reverse implication. ■

Proof 5.1:   Let f be a relative fitness function, differentiable at all $x \in S^n$. Let $\partial f_i(x)/\partial x_k > 0$ for all $k \neq i$, $x > \mathbf{0}$. Let $y \in S^n$ satisfy $y > \mathbf{0}$ and $f(y) = \mathbf{0}$.

Define for every $p \in \mathbb{R}^{n+1}$, $p > \mathbf{0}$, the excess demand function F by

$$F(p) = f(x(p)), \text{ where } x(p) \equiv p/(\Sigma_j p_j), j \in I^{n+1}.$$

For $p \in \mathbb{R}^{n+1}$, $p > \mathbf{0}$, F is continuous, $p^T F(p) = (\Sigma_j p_j) x(p)^T f(x(p)) = 0$, and F is positively homogeneous of degree zero, $F(\lambda p) = F(p)$ for any $\lambda > 0$.

F is differentiable, moreover $\partial F_i(p)/\partial p_k = (\Sigma_j p_j)^{-1} \partial f_i(x(p))/\partial x_k(p) > 0$.

By Lemma 5.1, it follows for an excess demand function fulfilling these conditions, that

(a)   there exists $q \in \mathbb{R}^{n+1}$, $q > \mathbf{0}$, satisfying $F(q) = \mathbf{0}$, and

(b)   if $F(p) = \mathbf{0}$ for some $p > \mathbf{0}$, then $p = \lambda q$ for some $\lambda > 0$.

Let $q > \mathbf{0}$ and $F(q) = \mathbf{0}$. Then $x(q) \in S^n$, and $F(q) = F(x(q)) = 0$.

Then $x(q) = y$, in view of b, which completes the proof. ■

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